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Amplitude Analysis of two-pseudoscalar systems

—Version VI—

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Abstract

The analytical tools needed for an amplitude analysis of the two-pseudoscalar systems are described in some detail.

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I. INTRODUCTION

This note describes in some detail the formalism necessary for exploring partial-wave amplitudes in a system involving two spinless particles. This note follows closely the notations and conventions spelled out previously by Chung [1] and Chung and Trueman [2].

The most familiar example of a system consisting of two pseudoscalar particles is that of $\pi\pi$, for which one must have $I + \ell$ =even [3]. If one restricts to the states with $\ell < 3$, then only P-wave is allowed for $I = 1$, whereas both S- and D-waves are possible if $I = 0$. Similarly, both S- and D-waves are allowed for an $\eta\eta$ system since $I = 0$ in this case. But for a system with two dissimilar spinless particles, e.g. $\pi\eta$ or $\eta\eta'$, all three possible states S, P and D should be present. It is for this reason that the $\pi\eta$ system has been chosen as an example in this note. The results presented here can be applied to the channels in which only the even waves allowed—by merely setting the P-wave amplitude and the odd moments to zero.

The angular distribution is specified uniquely, once a set of moments $\{H\}$ is given. For measurements of these moments from experimental data, it is advocated that the technique of the extended maximum-likelihood analysis [4] be used. The method detailed here for searching for ambiguities in the amplitudes follows and expands on that given by Sadovsky [5].

Sections II and III deal with the angular distributions and the amplitudes in the reflectivity basis. Section IV is devoted to the ambiguities in the partial waves, and an example is worked out in detail for a system consisting of S-, P- and D-waves in Section V. Experimental mesurements of the moments in the case of a finite acceptance and the extended likelihood function needed for finding a set of 'true' moments are given in Section VI. Finally,

in Section VII (Appendix) the d-functions, as well the e-functions derived from them, are given explicitly for ℓ up to 4.

It should be emphasized that the amplitude analysis on two-body spinless particles necessarily entails simplifying assumptions. These assumptions are not needed when a partial-wave analysis is carried out on three- and four-body final states. One fundamental reason for this is that the decay space expands from two dimensions for two-body to five and eight dimensions for three- and four-body systems.

One must conclude therefore that a more reliable spin-parity analysis of a resonance should be done on its three- and four-body decay modes.

II. GENERAL ANGULAR DISTRIBUTIONS

Consider the following reaction

$$\pi^- p \to \pi^0 \eta n \tag{1}$$

In the Jackson frame, the amplitudes may be expanded in terms of the partial waves for the $\pi\eta$ system:

$$U_k(\Omega) = \sum_{\ell m} V_{\ell m k} A_{\ell m}(\Omega) \tag{2}$$

where $V_{\ell m k}$ stands for the production amplitude for a state $|\ell m\rangle$ and k represents the spin degrees of freedom for the initial and final nucleons ($k = 1, 2$ for spin-flip and spin-nonflip amplitudes). $A_{\ell m}(\Omega)$ is the decay amplitude given by

$$A_{\ell m}(\Omega) = \sqrt{\frac{2\ell + 1}{4\pi}}\, D^{\ell\ *}_{m0}(\phi, \theta, 0) \tag{3}$$

where the angles $\Omega = (\theta, \phi)$ describe the direction of the η in the Jackson frame. The angular distribution is defined by

$$I(\Omega) = \sum_k |U_k(\Omega)|^2 \tag{4}$$

It should be emphasized that the nucleon helicities are external entities and the summation on k is applied to the absolute square of the amplitudes. A complete study of the $\pi\eta$ system requires four variables: $m(\pi\eta)$, $-t$ and the two angles in Ω. The distribution (4) is therefore to be applied to a given bin of $m(\pi\eta)$ and of $-t$.

The angular distribution may be expanded in terms of the moments $H(LM)$ via

$$I(\Omega) = \sum_{LM} \left(\frac{2L+1}{4\pi}\right) H(LM) D^{L\,*}_{M0}(\phi, \theta, 0) \tag{5}$$

where

$$H(LM) = \sum_{\substack{\ell m \\ \ell' m'}} \left(\frac{2\ell'+1}{2\ell+1}\right)^{1/2} \rho^{\ell\ell'}_{mm'} (\ell' m' LM | \ell m)(\ell' 0 L 0 | \ell 0) \tag{6}$$

where ρ is the spin-density matrix given by

$$\rho^{\ell\ell'}_{mm'} = \sum_k V_{\ell m k} V^*_{\ell' m' k} \tag{7}$$

It is seen that the moments $H(LM)$ are measurable quantities since

$$H(LM) = \int \mathrm{d}\Omega \; I(\Omega)\, D^L_{M0}(\phi, \theta, 0) \tag{8}$$

The normalization integral is

$$H(00) = \int \mathrm{d}\Omega \; I(\Omega) \tag{9}$$

The symmetry relations for H's are well known. From the hermiticity of ρ, one gets

$$H^*(LM) = (-)^M H(L-M) \tag{10}$$

and, from parity conservation in the production process, one finds

$$H(LM) = (-)^M H(L-M) \tag{11}$$

These show that H's are real. The angular distribution can now be recast into

$$I(\Omega) = \sum_{LM} \left(\frac{2L+1}{4\pi}\right) \tau(M) H(LM) d^L_{M0}(\theta) \cos M\phi \tag{12}$$

where

$$\begin{aligned} \tau(M) &= 2, \quad M > 0, \\ &= 1, \quad M = 0, \\ &= 0, \quad M < 0 \end{aligned} \tag{13}$$

Note that all the terms of (12) are now real. Since the D-functions form a complete orthonormal set in the space $\Omega = (\theta, \phi)$, one merely needs to specify a set of the H's to uniquely define an angular distribution.

Let ℓ_m be the maximum spin present in a given $\pi\eta$ mass bin. It is easy to show that the number of independent H's are

$$N_0 = (\ell_m + 1)(2\ell_m + 1) \tag{14}$$

N_0 as a function of ℓ_m is given below as a table:

ℓ_m	0	1	2	3	4
N_0	1	6	15	28	45

One makes a crucial assumption for amplitude analyses—the z-component m of spin ℓ can take on the values 0 or 1 only, i.e. the production amplitudes V are zero if $m > 1$. This implies that the H's are zero if $M > 2$. For $\ell_m > 0$ the number of zero H's is

$$N_1 = (\ell_m - 1)(2\ell_m - 1) \tag{15}$$

so that the number of non-zero moments is

$$N_2 = N_0 - N_1 = 6\ell_m \tag{16}$$

valid for $\ell_m > 0$. Now the number of non-zero H's are linear in ℓ_m. This allows one to determine amplitudes, as shown in the next section.

N_2 as a function of ℓ_m is tabulated below:

ℓ_m	0	1	2	3	4
N_2	1	6	12	18	24

III. AMPLITUDE ANALYSIS

The beginning point of an amplitude analysis is the reflection operator which preserves all the relevant momenta in the S-matrix and act directly on the rest-states of the particles involved in the production process. It is important to remember that the coordinate system is always defined with the y-axis along the production normal. In this case the reflection operator is simply the parity operator followed by a rotation by π around the y-axis.

The eigenstates of this reflection operator is

$$|\epsilon\ell m\rangle = \theta(m)\Big\{|\ell m\rangle - \epsilon(-)^m|\ell - m\rangle\Big\} \tag{17}$$

where

$$\begin{aligned}\theta(m) &= \frac{1}{\sqrt{2}}, \quad m > 0 \\ &= \frac{1}{2}, \quad m = 0 \\ &= 0, \quad m < 0\end{aligned} \tag{18}$$

One sees that $\tau(m) = 4\theta^2(m)$.

For a positive reflectivity, the $m = 0$ states are not allowed, i.e.

$$|\epsilon\ell 0\rangle = 0, \quad \text{if} \quad \epsilon = + \tag{19}$$

The reflectivity quantum number ϵ has been defined so that it coincides with the naturality of the exchanged particle in Reaction (1). One can prove this by noting that the meson production vertex is in reality a time-reversed process in which a state of arbitrary spin-parity decays into a pion (the beam) and a particle of a given naturality (the exchanged particle)

$$J^{\eta_J} \rightarrow s^{\eta_s} + \pi \tag{20}$$

where η's stand for intrinsic parities. The helicity-coupling amplitude F^J for this decay [1] is

$$A^J_p(M) \propto F^J_\lambda \; D^{J\,*}_{M\,\lambda}(\phi_p, \theta_p, 0) \tag{21}$$

where λ is the helicity of the exchanged particle and the subscript p stands for the 'production' variables. M is the z-component of spin J in a J rest frame. From the parity conservation in the decay, one finds

$$F^J_\lambda = -F^J_{-\lambda} \tag{22}$$

where one has used the relationships $\eta_J = (-)^J$ (true for two-pseudoscalar systems) and $\eta_s = (-)^s$ (natural-parity exchange). The formula shows that the helicity-coupling amplitude F^J is zero if λ is zero. Since angular momentum is conserved, its decay into two spinless particles in the final state cannot have $M = 0$ along the beam direction (the Jackson rest system), i.e. the D^J-function is zero unless $M = \lambda$, if $\theta_p = \phi_p = 0$. Finally, one may identify J with ℓ and M with m, which proves (19).

The modified D-functions in the reflectivity basis are given by

$${}^{\epsilon}D^{\ell\,*}_{m0}(\phi, \theta, 0) = \theta(m)\Big[D^{\ell\,*}_{m0}(\phi, \theta, 0) - \epsilon(-)^m \, D^{\ell\,*}_{-m0}(\phi, \theta, 0)\Big] \tag{23}$$

It is seen that the modified D-functions are real if $\epsilon = -1$ and imaginary if $\epsilon = +1$:

$$\begin{aligned} {}^{(-)}D^{\ell\,*}_{m0}(\phi, \theta, 0) &= 2\theta(m) d^{\ell}_{m0}(\theta) \cos m\phi \\ {}^{(+)}D^{\ell\,*}_{m0}(\phi, \theta, 0) &= 2i\theta(m) d^{\ell}_{m0}(\theta) \sin m\phi \end{aligned} \tag{24}$$

The overall amplitude in the reflectivity basis is now

$$^{\epsilon}U_k(\Omega) = \sum_{\ell m} {}^{\epsilon}V_{\ell m k}\, {}^{\epsilon}A_{\ell m}(\Omega) \tag{25}$$

where

$$^{\epsilon}A_{\ell m}(\Omega) = \sqrt{\frac{2\ell+1}{4\pi}}\, {}^{\epsilon}D^{\ell\,*}_{m0}(\phi,\theta,0) \tag{26}$$

and the resulting angular distribution is

$$I(\Omega) = \sum_{\epsilon k} |{}^{\epsilon}U_k(\Omega)|^2 \tag{27}$$

It is seen that the sum involves four non-interfering terms for $\epsilon = \pm$ and $k = 1, 2$. The absence of the interfering terms of different reflectivities is a direct consequence of parity conservation in the production process.

The moments $H(LM)$ can be expressed in terms of the amplitudes in the reflectivity basis. From the definition (8), one finds

$$H(LM) = \sum_{\epsilon} \sum_{\substack{\ell m \\ \ell' m'}} \left(\frac{2\ell'+1}{2\ell+1}\right)^{1/2} {}^{\epsilon}\rho^{\ell\ell'}_{mm'}\; {}^{\epsilon}b(\ell' m' L M \ell m)\; (\ell' 0 L 0 | \ell 0) \tag{28}$$

where ${}^{\epsilon}\rho$ is the spin-density matrix in the reflectivity basis given by

$$^{\epsilon}\rho^{\ell\ell'}_{mm'} = \sum_{k} {}^{\epsilon}V_{\ell m k}\, {}^{\epsilon}V^{*}_{\ell' m' k} \tag{29}$$

and a new function ${}^{\epsilon}b$ is a sum of Clebsch-Gordan coefficients:

$$\begin{aligned} {}^{\epsilon}b(\ell' m' L M \ell m) = \theta(m')\theta(m)\Big[&(\ell' m' L M | \ell m) + (-)^{M}\,(\ell' m' L\; -M | \ell m) \\ &-\epsilon\,(-)^{m'}\,(\ell'\; -m' L M | \ell m) - \epsilon\,(-)^{m}\,(\ell' m' L M | \ell\; -m)\Big] \end{aligned} \tag{30}$$

The formula (28) is useful in checking that a given set of the partial waves satisfies the experimentally measured moments $H(LM)$.

At this point one makes the second crucial assumption—necessary for carrying out the amplitude analysis: the production amplitudes ${}^{\epsilon}V$ do not depend on k. The distribution function is then given by a sum of two terms

$$I(\Omega) = |^{(+)}U(\Omega)|^2 + |^{(-)}U(\Omega)|^2 \tag{31}$$

The first assumption demands that the production amplitudes ${}^{\epsilon}V$ should be zero if $m > 1$. It is therefore convenient to separate out the θ dependence from that of ϕ, as follows:

$$\begin{aligned} {}^{(-)}U(\Omega) &= \frac{1}{\sqrt{4\pi}}\left[h_0(\theta) + \sqrt{2}h_-(\theta)\cos\phi\right] \\ {}^{(+)}U(\Omega) &= \frac{1}{\sqrt{4\pi}}\left[\sqrt{2}h_+(\theta)\sin\phi\right] \end{aligned} \tag{32}$$

where

$$\begin{aligned} h_0(\theta) &= \sum_{\ell=0}^{\ell_m}\sqrt{2\ell+1}\,[\,\ell\,]_0\, d^{\ell}_{00}(\theta) \\ h_-(\theta) &= \sum_{\ell=1}^{\ell_m}\sqrt{2\ell+1}\,[\,\ell\,]_-\, d^{\ell}_{10}(\theta) \\ h_+(\theta) &= \sum_{\ell=1}^{\ell_m}\sqrt{2\ell+1}\,[\,\ell\,]_+\, d^{\ell}_{10}(\theta) \end{aligned} \tag{33}$$

Note that

$$h_0(-\theta) = +h_0(\theta) \quad \text{and} \quad h_\pm(-\theta) = -h_\pm(\theta) \tag{34}$$

because of a symmetry relation for the d-functions [see (95) of Appendix]. Following convention, one has introduced a notation for partial amplitudes via

$$[\,\ell\,]_0 = {}^{(-)}V_{\ell 0}, \quad [\,\ell\,]_- = {}^{(-)}V_{\ell 1}, \quad [\,\ell\,]_+ = {}^{(+)}V_{\ell 1} \tag{35}$$

It is to be understood that the notation $[\,\ell\,]$ stands for the partial waves S, P, D, F and G for ℓ =0, 1, 2, 3 and 4.

It is useful to write down explicitly the formulas for $H(LM)$ in terms of the partial waves. From (28), one finds

$$\begin{aligned} H(L0) &= \sum_{\ell\ell'}\sqrt{\frac{2\ell'+1}{2\ell+1}}\,\Big\{[\ell]_0[\ell']_0^*\,(\ell'0L0|\ell0) + [\ell]_-[\ell']_-^*\,(\ell'1L0|\ell1) \\ &\qquad\qquad + [\ell]_+[\ell']_+^*\,(\ell'1L0|\ell1)\Big\}(\ell'0L0|\ell0) \\ H(L1) &= \frac{1}{\sqrt{2}}\sum_{\ell\ell'}\sqrt{\frac{2\ell'+1}{2\ell+1}}\,\Big\{[\ell]_-[\ell']_0^*\,(\ell'0L1|\ell1) - [\ell]_0[\ell']_-^*\,(\ell'\,-1L1|\ell0)\Big\}(\ell'0L0|\ell0) \\ H(L2) &= \frac{1}{2}\sum_{\ell\ell'}\sqrt{\frac{2\ell'+1}{2\ell+1}}\,\Big\{-[\ell]_-[\ell']_-^*\,(\ell'\,-1L2|\ell1) + [\ell]_+[\ell']_+^*\,(\ell'\,-1L2|\ell1)\Big\}(\ell'0L0|\ell0) \end{aligned} \tag{36}$$

These equations can be transformed further, as follows:

$$
\begin{aligned}
H(L0) &= \sum_{\ell}\Big\{[\ell]_0[\ell]_0^*\,(\ell 0L0|\ell 0) + [\ell]_-[\ell]_-^*\,(\ell 1L0|\ell 1) \\
&\qquad\qquad +[\ell]_+[\ell]_+^*\,(\ell 1L0|\ell 1)\Big\}(\ell 0L0|\ell 0) \\
&\quad +2\sum_{\ell<\ell'}\sqrt{\frac{2\ell'+1}{2\ell+1}}\,\mathrm{Re}\Big\{[\ell]_0[\ell']_0^*\,(\ell' 0L0|\ell 0) + [\ell]_-[\ell']_-^*\,(\ell' 1L0|\ell 1) \\
&\qquad\qquad +[\ell]_+[\ell']_+^*\,(\ell' 1L0|\ell 1)\Big\}(\ell' 0L0|\ell 0) \\
H(L1) &= \sqrt{2}\sum_{\ell}\mathrm{Re}\Big\{[\ell]_-[\ell]_0^*\Big\}(\ell 0L1|\ell 1)(\ell 0L0|\ell 0) \\
&\quad +\sqrt{2}\sum_{\ell<\ell'}\sqrt{\frac{2\ell'+1}{2\ell+1}}\,\mathrm{Re}\Big\{[\ell]_-[\ell']_0^*\,(\ell' 0L1|\ell 1) - [\ell]_0[\ell']_-^*\,(\ell' -1L1|\ell 0)\Big\}(\ell' 0L0|\ell 0) \qquad (37) \\
H(L2) &= \frac{1}{2}\sum_{\ell}\Big\{-[\ell]_-[\ell]_-^*\,(\ell -1L2|\ell 1) + [\ell]_+[\ell]_+^*\,(\ell -1L2|\ell 1)\Big\}(\ell 0L0|\ell 0) \\
&\quad +\sum_{\ell<\ell'}\sqrt{\frac{2\ell'+1}{2\ell+1}}\,\mathrm{Re}\Big\{-[\ell]_-[\ell']_-^*\,(\ell' -1L2|\ell 1) + [\ell]_+[\ell']_+^*\,(\ell' -1L2|\ell 1)\Big\}(\ell' 0L0|\ell 0)
\end{aligned}
$$

One sees that only the real part of the interference terms contribute to the moments and that the moments themselves are explicitly real. It is worth mentioning that the moments $H(L1)$ have contributions only from the waves with unnatural-parity exchange. Note also that the moments are zero identically unless $\ell'+\ell+L =$ even. The first formula above shows that $H(00)$ is simply a sum of all the waves in the problem, i.e.

$$
H(00) = \sum_{\ell}\Big\{|[\ell]_0|^2 + |[\ell]_-|^2 + |[\ell]_+|^2\Big\} \qquad (38)
$$

There are three amplitudes for each ℓ, except for the S-wave which comes with only one amplitude. The angular distribution is a sum of two non-interfering terms as given in (31), so that one amplitude each for $\epsilon = \pm$ can be set to be real. There are therefore $6\ell_m$ real parameters to be determined—exactly equal to the number N_2 of H's as given in the previous section. This allows one to solve in principle for the partial waves, given a set of the moments $\{H\}$. For example, one finds that, if $\ell_m = 2$, 12 non-zero moments are

$$
\begin{aligned}
&H(00),\ H(10),\ H(11),\ H(20),\ H(21),\ H(22) \\
&H(30),\ H(31),\ H(32),\ H(40),\ H(41),\ H(42) \qquad (39)
\end{aligned}
$$

while the partial waves $[\,\ell\,]$ are, for unnatural-parity exchange,

$$S_0, \quad P_0, \quad P_-, \quad D_0, \quad D_- \tag{40}$$

and, for natural-parity exchange,

$$P_+, \quad D_+ \tag{41}$$

One wave in each naturality can be set be real (S_0 and P_+, for example), so that there are again 12 real parameters to be determined.

IV. AMBIGUITIES IN THE PARTIAL WAVES

It is instructive to rewrite the angular distribution as follows:

$$I(\Omega) = \frac{1}{4\pi}\Big[f_0(\theta) + 2f_1(\theta)\cos\phi + 2f_2(\theta)\cos 2\phi\Big] \tag{42}$$

The f-functions are experimentally measurable, as they are completely determined given a set of moments $\{H\}$. Indeed one finds, from (12),

$$f_M(\theta) = \sum_{L=0}^{2\ell_m}(2L+1)H(LM)d^L_{M0}(\theta) \tag{43}$$

where ℓ_m is again the maximum ℓ in the problem. An alternative expression for $I(\Omega)$ as a function of the partial waves $[\,\ell\,]$ is, from (31) and (32),

$$I(\Omega) = \frac{1}{4\pi}\Big\{|h_0(\theta) + \sqrt{2}h_-(\theta)\cos\phi|^2 + |\sqrt{2}h_+(\theta)\sin\phi|^2\Big\} \tag{44}$$

Comparing the two expressions for $I(\Omega)$, one finds

$$\begin{aligned} f_0(\theta) &= |h_0(\theta)|^2 + |h_-(\theta)|^2 + |h_+(\theta)|^2 \\ f_1(\theta) &= \sqrt{2}\,\mathrm{Re}\{h_0(\theta)h^*_-(\theta)\} \\ f_2(\theta) &= \frac{1}{2}\Big\{|h_-(\theta)|^2 - |h_+(\theta)|^2\Big\} \end{aligned} \tag{45}$$

These equations succinctly summarizes the problem at hand; on the left-hand side are the functions involving measured H's, and one the right-hand side are the functions containing partial waves $[\,\ell\,]$ to be determined.

One may eliminate h_+ by combining $f_0(\theta)$ and $f_2(\theta)$ and modify $f_1(\theta)$ to obtain

$$\begin{aligned} f_a(\theta) &\equiv f_0(\theta) + 2f_2(\theta) = |h_0(\theta)|^2 + |\sqrt{2}\,h_-(\theta)|^2 \\ f_b(\theta) &\equiv 2f_1(\theta) = 2\mathrm{Re}\{h_0(\theta)\sqrt{2}\,h_-^*(\theta)\} \end{aligned} \tag{46}$$

The form of f_a and f_b suggests that one can define, from (34),

$$\begin{aligned} g(\theta) &= \frac{1}{\sqrt{2}}\Big[h_0(\theta) + \sqrt{2}\,h_-(\theta)\Big] \\ g(-\theta) &= \frac{1}{\sqrt{2}}\Big[h_0(\theta) - \sqrt{2}\,h_-(\theta)\Big] \end{aligned} \tag{47}$$

and find

$$\begin{aligned} f_a(\theta) &= |g(\theta)|^2 + |g(-\theta)|^2 \\ f_b(\theta) &= |g(\theta)|^2 - |g(-\theta)|^2 \end{aligned} \tag{48}$$

In order to examine the ambiguities in the problem, it is necessary to express the h-functions as ratios of polynomials in a single variable. This is accomplished by introducing a variable $u = \tan(\theta/2)$ and the functions $e^{\ell}_{m'm}(u)$ as shown in the Appendix. One finds that the h-functions assume the form,

$$\begin{aligned} h_0(u) &= \frac{\sum_{\ell=0}^{\ell_m}\sqrt{2\ell+1}\,[\,\ell\,]_0\,(1+u^2)^{\ell_m-\ell}\,e^{\ell}_{00}(u)}{(1+u^2)^{\ell_m}} \\ h_-(u) &= \frac{\sum_{\ell=1}^{\ell_m}\sqrt{2\ell+1}\,[\,\ell\,]_-\,(1+u^2)^{\ell_m-\ell}\,e^{\ell}_{10}(u)}{(1+u^2)^{\ell_m}} \\ h_+(u) &= \frac{\sum_{\ell=1}^{\ell_m}\sqrt{2\ell+1}\,[\,\ell\,]_+\,(1+u^2)^{\ell_m-\ell}\,e^{\ell}_{10}(u)}{(1+u^2)^{\ell_m}} \end{aligned} \tag{49}$$

It is seen that the numerator of $h_0(u)$ is a polynomial in u^2 of order ℓ_m. The definition (100) of Appendix shows that the numerator of $h_\pm(u)$ is expressed as u times a polynomial in u^2

of order $\ell_m - 1$. Consequently, the function $g(u)$ has a numerator expressed as a polynomial in u of order $2\ell_m$. The functions f_M may also be given in terms of the e-functions:

$$f_M(u) = \frac{\sum_{L=0}^{2\ell_m}(2L+1)H(LM)(1+u^2)^{2\ell_m - L}\, e^L_{M0}(u)}{(1+u^2)^{2\ell_m}} \tag{50}$$

Suppose now that a set of $[\,\ell\,]$ has been found satisfying (45). One can then find $2\ell_m$ roots of the function

$$(1+u^2)^{\ell_m}\, g(u) = c_0 \prod_{k=1}^{2\ell_m}(u - u_k) \tag{51}$$

where u_k's are complex roots—these are the so-called 'Barrelet' zeroes [6]—and c_0 is a complex constant. Since $g(u)$ and $g(-u)$ enter as absolute squares in the expression for the f-functions [see (48)], the complex conjugate of a root u_k is an equally valid solution, i.e. the f-functions are not perturbed. One must therefore conclude that, for the partial waves corresponding to unnatural-parity exchange ($\epsilon = -$), there are in general $2^{2\ell_m - 1}$ solutions to the problem, after eliminating those which may be obtained by taking complex conjugation of the entire function $g(u)$. For each new $g(u)$, one may calculate

$$\begin{aligned} h_0(\theta) &= \frac{1}{\sqrt{2}}\big[g(\theta) + g(-\theta)\big] \\ h_-(\theta) &= \frac{1}{2}\big[g(\theta) - g(-\theta)\big] \end{aligned} \tag{52}$$

to search for a new set of partial waves $[\,\ell\,]_0$ and $[\,\ell\,]_-$. Next, the third equation of (45) is used to calculate

$$|h_+(\theta)|^2 = |h_-(\theta)|^2 - 2f_2(\theta) \tag{53}$$

Note that all the allowed set of partial waves, $[\,\ell\,]_0$ and $[\,\ell\,]_-$, must satisfy the condition that the right-hand side of this equation remain non-negative. The first equation of (45), of course, does not constitute a new constraint. The relationship (53) also indicates that the ambiguity problem for $[\,\ell\,]_+$ can be dealt with by setting

$$(1+u^2)^{\ell_m}\, h_+(u) = c_+\, u \prod_{k=1}^{\ell_m - 1}(u^2 - r_k) \tag{54}$$

where r_k's are the complex roots in u^2 and c_+ is a complex constant. For $\ell_m > 1$, there must be in general $2^{\ell_m - 2}$ solutions for the partial waves with natural-parity exchange, i.e. $[\ell]_+$.

For each of the $2^{2\ell_m - 1}$ solutions involving $[\ell]_0$ and $[\ell]_-$, there must be a total of $2^{\ell_m - 2}$ solutions for $[\ell]_+$. Therefore, a system containing the partial waves up to $\ell = \ell_m \geq 2$ has a total of $2^{2\ell_m - 1} \times 2^{\ell_m - 2}$ ambiguous solutions. In the next two sections are given two illustrative, but practically important, examples.

V. AN EXAMPLE WITH S-, P- AND D-WAVES

Consider once again an example with $\ell_m = 2$. For the sake of completeness, all the relevant moments are given in terms of the partial waves in the problem, see (37). The results are tabulated in Tables Ia and Ib. It should be noted that a moment is in fact a sum of all the terms in Tables Ia and Ib.

Table Ia. Moments in terms of Partial Waves

	U [a] (00)(00)	U (00)(10)	U (00)(11)	U (00)(20)	U (00)(21)	U (10)(10)	U (10)(11)	U (10)(20)	U (10)(21)
$H(00)$	1					1			
$H(10)$		$1/3$ [b]						$4/15$	
$H(11)$			$1/6$						$1/10$
$H(20)$				$1/5$		$4/25$			
$H(21)$					$1/10$		$3/50$		
$H(30)$								$27/245$	
$H(31)$									$12/245$

[a] The waves with unnatural-parity exchange. The notations are, e.g.

$(00)(00) = S_0\, S_0^* = |S_0|^2$ and $(00)(11) = S_0\, P_-^* + P_-\, S_0^* = 2\mathrm{Re}\{S_0\, P_-^*\}$.

[b] A square root is understood for both numerators and denominators.

An example: $-2/3$ stands for $-\sqrt{2/3}$.

Table Ib. Moments in terms of Partial Waves

	U [a]	U	U	U	U	U	N [b]	N	N
	(11)(11)	(11)(20)	(11)(21)	(20)(20)	(20)(21)	(21)(21)	(11)(11)	(11)(21)	(21)(21)
$H(00)$	1			1		1	1		1
$H(10)$			$1/5$ [c]					$1/5$	
$H(11)$		$-1/30$							
$H(20)$	$-1/25$			$4/49$		$1/49$	$-1/25$		$1/49$
$H(21)$					$1/98$				
$H(22)$	$3/50$					$3/98$	$-3/50$		$-3/98$
$H(30)$			$-9/245$					$-9/245$	
$H(31)$		$9/245$							
$H(32)$			$3/98$					$-3/98$	
$H(40)$				$4/49$		$-16/441$			$-16/441$
$H(41)$					$5/147$				
$H(42)$						$10/441$			$-10/441$

[a] The waves with unnatural-parity exchange. The notations are, e.g.

$(20)(20) = D_0\, D_0^* = |D_0|^2$ and $(11)(21) = P_-\, D_-^* + D_-\, P_-^* = 2\mathrm{Re}\{P_-\, D_-^*\}$.

[b] The waves with natural-parity exchange. The notations are, e.g.

$(11)(11) = P_+\, P_+^* = |P_+|^2$ and $(11)(21) = P_+\, D_+^* + D_+\, P_+^* = 2\mathrm{Re}\{P_+\, D_+^*\}$.

[c] A square root is understood for both numerators and denominators.

An example: $-2/3$ stands for $-\sqrt{2/3}$.

One should note that the moments $H(4M)$ have contributions from the D-wave only, while the moments $H(3M)$ result from an interference between P- and D-waves.

In terms of the experimentally measured moments $\{H\}$'s, the f-functions are, from (43),

$$(1+u^2)^4\, f_M(u) = \sum_{L=0}^{4} (2L+1) H(LM) (1+u^2)^{4-L}\, e^L_{M0}(u) \tag{55}$$

or, explicitly,

$$
\begin{aligned}
(1+u^2)^4\, f_0(u) = {} & H(00)(1+u^2)^4 + 3H(10)(1+u^2)^3\,(1-u^2) \\
& +5H(20)(1+u^2)^2\,(1-4u^2+u^4) \\
& +7H(30)(1+u^2)\,(1-9u^2+9u^4-u^6) \\
& +9H(40)(1-16u^2+36u^4-16u^6+u^8) \\
(1+u^2)^4\, f_1(u) = {} & -3\sqrt{2}H(11)(1+u^2)^3 u \\
& -5\sqrt{6}H(21)(1+u^2)^2 u(1-u^2) \\
& -14\sqrt{3}H(31)(1+u^2)u(1-3u^2+u^4) \\
& -18\sqrt{5}H(41)u(1-6u^2+6u^4-u^6) \\
(1+u^2)^4\, f_2(u) = {} & 5\sqrt{6}H(22)(1+u^2)^2 u^2 \\
& +7\sqrt{30}H(32)(1+u^2)u^2(1-u^2) \\
& +9\sqrt{10}H(42)u^2(3-8u^2+3u^4)
\end{aligned}
\tag{56}
$$

Suppose now that one has found a set of solutions $\{S_0,\, P_0,\, P_-,\, D_0,\, D_-\}$ for unnatural-parity exchange and $\{P_+,\, D_+\}$ for natural-parity exchange. It is helpful to write out explicitly the formulas for h's:

$$
\begin{aligned}
(1+u^2)^2\, h_0(u) = {} & S_0\,(1+u^2)^2 + \sqrt{3}P_0\,(1-u^4) + \sqrt{5}D_0\,(1-4u^2+u^4) \\
\sqrt{2}(1+u^2)^2\, h_-(u) = {} & -2u\Big[\sqrt{3}P_-\,(1+u^2) + \sqrt{15}D_-\,(1-u^2)\Big] \\
\sqrt{2}(1+u^2)^2\, h_+(u) = {} & -2u\Big[\sqrt{3}P_+\,(1+u^2) + \sqrt{15}D_+\,(1-u^2)\Big]
\end{aligned}
\tag{57}
$$

The last equation above shows that there are no ambiguities for the partial waves P_+ and D_+, since the expression inside the square bracket is linear in u^2. On the other hand, from the first two equations, one finds that the function $g(u)$ is given by

$$
\begin{aligned}
G(u) \equiv \sqrt{2}(1+u^2)^2\, g(u) = {} & S_0\,(1+u^2)^2 + \sqrt{3}P_0\,(1-u^4) + \sqrt{5}D_0\,(1-4u^2+u^4) \\
& -2\sqrt{3}P_-\,(u+u^3) - 2\sqrt{15}D_-\,(u-u^3)
\end{aligned}
\tag{58}
$$

which is a polynomial of order 4 in u and thus gives rise to the ambiguities in the unnatural-parity partial-waves through the Barrelet zeroes.

One may write it as

$$G(u) = a_4\, u^4 - a_3\, u^3 + a_2\, u^2 - a_1\, u + a_0 \tag{59}$$

with

$$\begin{aligned} a_4 &= S_0 - \sqrt{3}P_0 + \sqrt{5}D_0 \\ a_3 &= 2\sqrt{3}(P_- - \sqrt{5}D_-) \\ a_2 &= 2S_0 - 4\sqrt{5}D_0 \\ a_1 &= 2\sqrt{3}(P_- + \sqrt{5}D_-) \\ a_0 &= S_0 + \sqrt{3}P_0 + \sqrt{5}D_0 \end{aligned} \tag{60}$$

The inverse is

$$\begin{aligned} 6S_0 &= 2a_0 + a_2 + 2a_4 \\ 2\sqrt{3}P_0 &= a_0 - a_4 \\ 6\sqrt{5}D_0 &= a_0 - a_2 + a_4 \\ 4\sqrt{3}P_- &= a_1 + a_3 \\ 4\sqrt{15}D_- &= a_1 - a_3 \end{aligned} \tag{61}$$

Since $G(u)$ is a 4th-order polynomial in u with 4 complex roots $\{u_1, u_2, u_3, u_4\}$, it could be given by

$$G(u) = a_4(u - u_1)(u - u_2)(u - u_3)(u - u_4) \tag{62}$$

so that

$$\begin{aligned} a_3 &= a_4(u_1 + u_2 + u_3 + u_4) \\ a_2 &= a_4(u_1u_2 + u_1u_3 + u_1u_4 + u_2u_3 + u_2u_4 + u_3u_4) \end{aligned} \tag{63}$$

$$a_1 = a_4(u_1u_2u_3 + u_2u_3u_4 + u_3u_4u_1 + u_4u_1u_2)$$

$$a_0 = a_4(u_1u_2u_3u_4)$$

Finally, substituting these into (61), the partial waves can be expressed in terms of the roots or the Barrelet zeroes:

$$\begin{aligned}
6S_0 &= a_4(2u_1u_2u_3u_4 + u_1u_2 + u_1u_3 + u_1u_4 + u_2u_3 + u_2u_4 + u_3u_4 + 2) \\
2\sqrt{3}P_0 &= a_4(u_1u_2u_3u_4 - 1) \\
6\sqrt{5}D_0 &= a_4(u_1u_2u_3u_4 - u_1u_2 - u_1u_3 - u_1u_4 - u_2u_3 - u_2u_4 - u_3u_4 + 1) \\
4\sqrt{3}P_- &= a_4(u_1u_2u_3 + u_2u_3u_4 + u_3u_4u_1 + u_4u_1u_2 + u_1 + u_2 + u_3 + u_4) \\
4\sqrt{15}D_- &= a_4(u_1u_2u_3 + u_2u_3u_4 + u_3u_4u_1 + u_4u_1u_2 - u_1 - u_2 - u_3 - u_4)
\end{aligned} \tag{64}$$

These expressions show how the ambiguities in the partial waves could be explored through complex conjugation of one or more of the Barrelet zeroes. It should be borne in mind, however, that a_4 above is fixed and is given by the original set of the partial waves, see (60). More precisely, its magnitude $|a_4|$ must remain invariant.

There should be in general 8 ambiguous solutions involving the partial waves S_0, P_0, P_-, D_0 and D_-. The 8 solutions are enumerated below in two columns:

$$\begin{array}{ll} \{u_1, u_2, u_3, u_4\} & \{u_1, u_2, u_3, u_4\} \\ \{u_1, u_2, u_3, u_4^*\} & \{u_1^*, u_2, u_3, u_4\} \\ \{u_1, u_2, u_3^*, u_4\} & \{u_1, u_2^*, u_3, u_4\} \\ \{u_1, u_2, u_3^*, u_4^*\} & \{u_1, u_2, u_3^*, u_4\} \\ \{u_1, u_2^*, u_3, u_4\} & \{u_1, u_2, u_3, u_4^*\} \\ \{u_1, u_2^*, u_3, u_4^*\} & \{u_1^*, u_2^*, u_3, u_4\} \\ \{u_1, u_2^*, u_3^*, u_4\} & \{u_1^*, u_2, u_3^*, u_4\} \\ \{u_1, u_2^*, u_3^*, u_4^*\} & \{u_1^*, u_2, u_3, u_4^*\} \end{array} \tag{65}$$

The first column results from a procedure in which u_1 is left invariant and the remaining three roots u_2, u_3 and u_4 are allowed to undergo complex conjugation—one sees that there are 2^3=8 ways of doing this. The second column represents an alternative method of ennumerating the 8 ambiguous solutions. For each solution, a new $G(u)$ is given by (62), and hence one obtains new h's via

$$\begin{aligned} h_0(u) &= \frac{1}{\sqrt{2}}\Big[g(u) + g(-u)\Big] \\ h_-(u) &= \frac{1}{2}\Big[g(u) - g(-u)\Big] \end{aligned} \tag{66}$$

and the new partial waves $\{S_0, P_0, P_-, D_0, D_-\}$ are given by (64). Note that S_0 may become complex in the process, but it can be made real again by dividing each wave in the set by the phase of S_0.

One can find the corresponding P_+ and D_+ from

$$|h_+(u)|^2 = |h_-(u)|^2 - 2f_2(u) \tag{67}$$

Formulas (56) and (57) show that, if the factor u^2 is taken out from both sides of the equation above, then one is left with a quadratic function in u^2. One can, therefore, find P_+, D_+ and

the phase difference between the two, by setting the three coefficients to zero. But a more direct way of finding them is contained in Tables Ia and Ib. One sees that

$$\begin{aligned} |D_+|^2 &= |D_-|^2 - \frac{21}{\sqrt{10}} H(42) \\ |P_+|^2 &= |P_-|^2 + 3\sqrt{\frac{5}{2}} H(42) - 5\sqrt{\frac{2}{3}} H(22) \\ 2\mathrm{Re}\{P_+ D_+^*\} &= 2\mathrm{Re}\{P_- D_-^*\} - 7\sqrt{\frac{2}{3}} H(32) \end{aligned} \tag{68}$$

It is clear that any solution resulting in negative values for the first two equations above is not allowed. In addition, any solution which makes the cosine of the phase difference greater than 1 is clearly not allowed.

VI. A SYSTEM WITH S-, P-, D- AND F-WAVES

In this section, a brief description is given of a system containing the partial waves $\ell = 0$, 1, 2 and 3. There are 13 parameters to be determined describing the seven amplitudes S_0, P_0, D_0, F_0, P_-, D_- and F_- produced via unnatural-parity exchange, and there are 5 parameters for three amplitudes P_+, D_+ and F_+ with natural-parity exchange. There are, on the other hand, 18 real moments $H(LM)$ with L=0–6 and M=0, 1 and 2, which specify the angular distribution for this problem. Therefore, it is in general possible to find a set of amplitudes satisfying the moments, but the process is not unique, and there are in fact a total of 64 ambiguous solutions.

The h-functions are given by, from Appendix,

$$\begin{aligned} (1+u^2)^3\, h_0(u) &= S_0\,(1+u^2)^3 + \sqrt{3} P_0\,(1+u^2)^2(1-u^2) \\ &\quad + \sqrt{5} D_0\,(1+u^2)(1-4u^2+u^4) + \sqrt{7} F_0\,(1-9u^2+9u^4-u^6) \\ \sqrt{2}(1+u^2)^3\, h_-(u) &= -2u\Big[\sqrt{3} P_-\,(1+u^2)^2 + \sqrt{15} D_-\,(1-u^4) \\ &\quad + \sqrt{42} F_-(1-3u^2+u^4)\Big] \end{aligned} \tag{69}$$

$$\sqrt{2}(1+u^2)^3\,h_+(u) = -2u\Big[\sqrt{3}P_+\,(1+u^2)^2 + \sqrt{15}D_+\,(1-u^4) + \sqrt{42}F_+(1-3u^2+u^4)\Big]$$

The function $g(u)$ is given by

$$G(u) \equiv \sqrt{2}(1+u^2)^2\,g(u) = a_6\,u^6 - a_5\,u^5 + a_4\,u^4 - a_3\,u^3 + a_2\,u^2 - a_1\,u + a_0 \tag{70}$$

which is a polynomial of order 6 in u and the ambiguities in the unnatural-parity partial-waves can be found through the Barrelet zeroes, as before. From (69), one sees that

$$\begin{aligned}
a_6 &= S_0 - \sqrt{3}P_0 + \sqrt{5}D_0 - \sqrt{7}F_0 \\
a_0 &= S_0 + \sqrt{3}P_0 + \sqrt{5}D_0 + \sqrt{7}F_0 \\
a_4 &= 3S_0 - \sqrt{3}P_0 - 3\sqrt{5}D_0 + 9\sqrt{7}F_0 \\
a_2 &= 3S_0 + \sqrt{3}P_0 - 3\sqrt{5}D_0 - 9\sqrt{7}F_0 \\
a_5 &= 2\sqrt{3}P_- - 2\sqrt{15}D_- + 2\sqrt{42}F_- \\
a_1 &= 2\sqrt{3}P_- + 2\sqrt{15}D_- + 2\sqrt{42}F_- \\
a_3 &= 4\sqrt{3}P_- - 6\sqrt{42}F_-
\end{aligned} \tag{71}$$

The inverse is

$$\begin{aligned}
12S_0 &= 3(a_6+a_0) + (a_4+a_2) \\
12\sqrt{5}D_0 &= 3(a_6+a_0) - (a_4+a_2) \\
20\sqrt{3}P_0 &= 9(a_0-a_6) + (a_2-a_4) \\
20\sqrt{7}F_0 &= (a_0-a_6) - (a_2-a_4) \\
20\sqrt{3}P_- &= 3(a_5+a_1) + 2a_3 \\
10\sqrt{42}F_- &= (a_5+a_1) - a_3 \\
4\sqrt{15}D_- &= a_1 - a_5
\end{aligned} \tag{72}$$

Since $G(u)$ is a 6th-order polynomial in u with 6 complex roots $\{u_i\}$, $i = 1$–6, it can be written

$$G(u) = a_6(u-u_1)(u-u_2)(u-u_3)(u-u_4)(u-u_5)(u-u_6) \tag{73}$$

so that

$$\begin{aligned}
a_5 &= a_6(u_1+u_2+u_3+u_4+u_5+u_6) \\
a_4 &= a_6 \sum_P u_i\, u_j \\
a_3 &= a_6 \sum_P u_i\, u_j\, u_k \\
a_2 &= a_6 \sum_P u_i\, u_j\, u_k\, u_\ell \\
a_1 &= a_6 \sum_P u_i\, u_j\, u_k\, u_\ell\, u_m \\
a_0 &= a_6(u_1u_2u_3u_4u_5u_6)
\end{aligned} \tag{74}$$

The sums are over all permutations with the condition $i<j<k<\ell<m$. Note that there are 15 terms in a_4, 20 in a_3, 15 in a_2 and 6 in a_1. Finally, substituting these into (72), the partial waves can be expressed in terms of the Barrelet zeroes. An operation $u_i \to u_i^*$ for any i does not perturb $H(LM)$ and therefore results in an ambiguous solution. Since taking the complex conjugate of all 6 roots is not a new solution, one can leave u_1 invariant, and take complex conjugates of all the rest—there are thus 2^5=32 ways of doing this, and this is the number of ambiguous solutions involving the waves with unnatural-parity exchange.

An examination of $h_+(u)$ in (69) reveals that it is proportional to a polynomial of order 2 in u^2 involving the partial waves with natural-parity exchange. Define $Q(u)$ by

$$\begin{aligned}
Q(u) &= \sqrt{3}P_+\,(1+u^2)^2 + \sqrt{15}D_+\,(1-u^4) + \sqrt{42}F_+(1-3u^2+u^4) \\
&= b_2u^4 - b_1u^2 + b_0
\end{aligned} \tag{75}$$

so that

$$\begin{aligned}
b_2 &= \sqrt{3}P_+ - \sqrt{15}D_+ + \sqrt{42}F_+ \\
b_0 &= \sqrt{3}P_+ + \sqrt{15}D_+ + \sqrt{42}F_+ \\
b_1 &= -2\sqrt{3}P_+ + 3\sqrt{42}F_+
\end{aligned} \tag{76}$$

The inverse is

$$
\begin{aligned}
5\sqrt{42}F_+ &= b_0 + b_1 + b_2 \\
10\sqrt{3}P_+ &= 3b_0 - 2b_1 + 3b_2 \\
2\sqrt{15}D_+ &= b_0 - b_2
\end{aligned}
\tag{77}
$$

Let $r_1$and r_2 be two roots given by

$$
Q(u) = b_2(u^2 - r_1)(u^2 - r_2) \tag{78}
$$

Then one finds

$$
\begin{aligned}
b_1 &= b_2(r_1 + r_2) \\
b_0 &= b_2 r_2 r_2
\end{aligned}
\tag{79}
$$

Finally, substituting these into (77) with $r_2 \rightarrow r_2^*$, one can obtain an alternate solution involving the partial waves with natural-parity exchange.

It should be noted that there exist two solutions involving P_+, D_+ and F_+ for each of the 32 partial-wave sets with unnatural-parity exchange—one sees therefore that there must exist in general a total of 64 ambiguous solutions for a system containing spins up to 3. Because there should exist in general two distinct solutions for P_+, D_+ and F_+, it is not possible to invert the formulas for $H(L2)$ with L=2–6 and solve for the the amplitudes algebraically—the reader will recall that this was possible for the problem containing ℓ =0, 1 and 2 (see Section V). Instead, one must resort to a function-minimization routine to find one set of solutions for P_+, D_+ and F_+ for each of the 32 partial-wave sets with unnatural-parity exchange; the second set can then be found through the the technique of the Barrelet zeroes [see (78)]. Not all 64 solutions are 'correct' in general; for each set one must calculate all the predicted $H(LM)$'s and check that they are identical to the experimental moments—at least within the error bars.

VII. EXPERIMENTAL MEASUREMENT OF THE MOMENTS

It is suggested that a set of the partial waves $[\ell]_0$, $[\ell]_-$ and $[\ell]_+$ be used directly as unknown parameters in the extended maximum-likelihood fits [4]. Since there is an absolute scale in an extended maximum-likelihood fit, one has then in possession the predicted numbers of events for all the partial waves, corrected for finite acceptance and angular distributions. The partial waves in turn give rise to a set of predicted moments $\{H\}$, see Tables Ia and Ib. But the $H(00)$ is not 1 [(9)] but the total predicted number of events from the fit [see (38)], i.e. one should be using the unnormalized moments. It is possible to choose H's as unknowns in the fit, but the two sets of H's should be the same ideally—this affords one an effective way of assessing self-consistency between the chosen moments and the partial waves.

One may determine directly the experimental moments (unnormalized) as follows:

$$H_x(LM) = \sum_i^n D^L_{M\,0}(\phi_i, \theta_i, 0) \tag{80}$$

where the sum is over a given number n of experimental data in a mass bin. But this is given by, from (8),

$$H_x(LM) = \int \mathrm{d}\Omega\ \eta(\Omega)\, I(\Omega)\, D^L_{M0}(\phi, \theta, 0) \tag{81}$$

where $\eta(\Omega)$ represents the finite acceptance of the apparatus, and it includes software cuts, if any. From (5), one finds that

$$H_x(LM) = \sum_{L'M'} H(L'M')\ \Psi_x(LM\ L'M') \tag{82}$$

where

$$\Psi_x(LM\ L'M') = \left(\frac{2L'+1}{4\pi}\right) \int \mathrm{d}\Omega\ \eta(\Omega)\ D^L_{M0}(\phi, \theta, 0)\, D^{L'\,*}_{M'0}(\phi, \theta, 0) \tag{83}$$

Note that the Ψ's have a simple normalization

$$\Psi_x(LM\,L'M') = \delta_{L\,L'}\delta_{M\,M'} \tag{84}$$

in the limit $\eta(\Omega) = 1$. The integral (83) can be calculated using a sample of 'accepted' MC events. Let N_x be the number of accepted MC events, out of a total of N raw MC events. Then, the integral is

$$\Psi_x(LM\,L'M') = \left(\frac{2L'+1}{4\pi}\right)\ \frac{1}{N}\sum_i^{N_x}\ D^L_{M0}(\phi_i,\theta_i,0)\,D^{L'}_{M'0}{}^*(\phi_i,\theta_i,0) \tag{85}$$

The equation (82) shows that one can predict the experimentally measurable moments (80), given a set $\{H\}$ and Ψ's; this provides one a means of assessing the goodness of fit by forming a χ^2 based on the set $\{H_x\}$.

There exists an alternative method of determining Ψ's. For the purpose, one expands the acceptance function $\eta(\Omega)$ in terms of the orthonormal D-functions, as follows:

$$\eta(\Omega) = \sum_{LM}(2L+1)\xi(LM)\,D^L_{M0}{}^*(\phi,\theta,0) \tag{86}$$

where $\xi(LM)$ is given by

$$\xi(LM) = \frac{1}{4\pi}\int \mathrm{d}\Omega\ \eta(\Omega)\,D^L_{M0}(\phi,\theta,0) \tag{87}$$

The complex conjugate is, from the defining formula above,

$$\xi^*(LM) = (-)^M\,\xi(L\ -M) \tag{88}$$

so that the acceptance function can be made explicitly real

$$\eta(\Omega) = \sum_{LM}(2L+1)\tau(M)\mathrm{Re}\left\{\xi(LM)\,D^L_{M0}{}^*(\phi,\theta,0)\right\} \tag{89}$$

using the $\tau(M)$ introduced in (13). A set of $\xi(LM)$ specifies completely the acceptance in the problem. The normalization for the acceptance function has be chosen so that a perfect acceptance is given by $\eta(\Omega) = 1$ and $\xi(LM) = \delta_{L0}\delta_{M0}$. The $\xi(LM)$'s can be measured experimentally using the accepted MC events

$$\xi(LM) = \frac{1}{4\pi N}\sum_i^{N_x}\ D^L_{M0}(\phi_i,\theta_i,0) \tag{90}$$

Finally, substituting (86) into (83), one finds

$$\Psi_x(LM\,L'M') = \sum_{L''M''} (2L''+1)\xi^*(L''M'')(LM\,L''M''|L'M')(L0L''0|L'0) \tag{91}$$

This formula shows an important aspect of the $\xi(LM)$ technique of representing acceptance. Although (87) involves a sum in which L and M could be extended to infinity for an arbitrary acceptance, there is a cutoff if the set $\{H\}$ has maxima L_m and M_m [see (82)]. The formula above demonstrates that $L'' \leq 2L_m$ and $|M''| \leq 2M_m$.

For completeness, the log of the extended likelihood function is given here. It can be written, for a sample of n experimental data in a given mass bin,

$$\ln\mathcal{L} \propto \sum_i^n \ln I(\Omega_i) - \int \mathrm{d}\Omega\ \eta(\Omega)\, I(\Omega) \tag{92}$$

which can be recast in terms of the $\xi(LM)$'s

$$\begin{aligned} \ln\mathcal{L} &\propto \sum_i^n \ln I(\Omega_i) - \sum_{LM} (2L+1)\, H(LM)\, \xi^*(LM) \\ &\propto \sum_i^n \ln I(\Omega_i) - \sum_{LM} (2L+1)\, \tau(M)\, H(LM)\, \mathrm{Re}\xi(LM) \end{aligned} \tag{93}$$

$H(LM)$'s may be used directly as parameters in the fit or may be given as functions of the partial waves. It is interesting to note that the $\xi(LM)$'s for $L > L_m$ and $|M| > M_m$ are not needed in the likelihood fit. Note also that only the real part of the $\xi(LM)$'s are used in the fit.

It should be borne in mind that a set of the moments $\{H\}$ may not always be expressed in terms of the partial waves. This is clear if one examines Tables Ia and Ib. Consider, for example, an angular distribution in which $H(10)$ is the only non-zero moment. But this moment is given by a set of interference terms involving even-odd partial waves. So at least one term cannot be zero—for example, the interference term involving S- and P-waves. But then neither $H(00)$ nor $H(20)$ can be zero, since both S- and P-waves are non-zero. One must conclude then that a χ^2 based on the set $\{H_x\}$ may not necessarily be zero identically.

VIII. APPENDIX

The d-functions are normally given, from Rose [7],

$$d^{\ell}_{m'm}(\theta) = (-)^{m'-m}[(\ell+m')!(\ell-m')!(\ell+m)!(\ell-m)!]^{1/2} \times \sum_{k=k_1}^{k_2} \frac{(-)^k \cos^{2\ell-n}(\theta/2)\sin^n(\theta/2)}{(\ell-m'-k)!(\ell+m-k)!(m'-m+k)!k!} \tag{94}$$

where $n = m' - m + 2k$ and k is a non-negative integer ranging between k_1 and k_2 given by

$$k_1 = \max\{0, m-m'\} \qquad k_2 = \min\{\ell-m', \ell+m\}$$

Note that n is also a non-negative integer. It is seen that the d-functions are non-singular polynomials of order up to 2ℓ in two variables $\cos(\theta/2)$ and $\sin(\theta/2)$; they both range from 0 to 1, as θ goes from 0 to π. From the definition (94), one finds the following relationships:

$$\begin{aligned} d^{\ell}_{m'm}(-\theta) &= (-)^{m'-m}\, d^{\ell}_{m'm}(\theta) \\ d^{\ell}_{m'm}(\theta) &= (-)^{m'-m}\, d^{\ell}_{mm'}(\theta) \\ d^{\ell}_{m'm}(\theta) &= (-)^{m'-m}\, d^{\ell}_{-m'-m}(\theta) \\ d^{\ell}_{m'm}(\pi-\theta) &= (-)^{\ell+m'}\, d^{\ell}_{m'-m}(\theta) \end{aligned} \tag{95}$$

The d-functions depend in reality on $z = \cos\theta$ only, through

$$x = \sin\frac{\theta}{2} = \sqrt{\frac{1-z}{2}} \quad \text{and} \quad y = \cos\frac{\theta}{2} = \sqrt{\frac{1+z}{2}} \tag{96}$$

This assertion is valid as long as θ remains within the range $(0 \to \pi)$. In particular, negative values of θ are not allowed in this scheme. Note that $\theta < 0$ is equivalent to

$$\Omega = (-\theta, \phi) \to \Omega' = (+\theta, \pi+\phi) \tag{97}$$

It should be remembered that θ is always positive in practice, since $\cos\theta$ is evaluated as a scalar product of two momenta. With this proviso then, there should not be any sign ambiguities in x and y as defined by (96). The d-functions in terms of the variables x and

y are given in this Appendix, as the existing tabulations are rarely given in these variables. For the purpose it is sufficient to consider only those with $m' \geq 0$ and $m' \geq |m|$ because of the symmetry relations for d [see (95)]. With these restrictions, the summation on k in (94) ranges from $k_1 = 0$ to $k_2 = \ell - m'$, and the d-functions take on the form

$$d^{\ell}_{m'm}(z) = (-)^{m'-m}[(\ell+m')!(\ell-m')!(\ell+m)!(\ell-m)!]^{1/2} \times x^{m'-m} \left[\sum_{k=k_1}^{k_2} \frac{(-)^k x^{2k} y^{2(\ell-m'-k)}}{(\ell-m'-k)!(\ell+m-k)!(m'-m+k)!k!}\right] y^{m'+m} \tag{98}$$

Note that all the exponents are non-negative. The function in the square bracket is a

polynomial in x^2 and in y^2, each of order $\ell - m'$. The d-function itself is a polynomial in x of order $2\ell - (m' + m)$ and in y of order $2\ell - (m' - m)$.

The d-functions are tabulated here for ℓ up to 3:

ℓ	m'	m	$d^{\ell}_{m'm}(z)$
0	0	0	1
1	1	1	y^2
1	1	0	$-\sqrt{2}\,xy$
1	1	-1	x^2
1	0	0	$y^2 - x^2$
2	2	2	y^4
2	2	1	$-2xy^3$
2	2	0	$\sqrt{6}\,x^2y^2$
2	2	-1	$-2x^3y$
2	2	-2	x^4
2	1	1	$(y^2 - 3x^2)y^2$
2	1	0	$-\sqrt{6}\,x(y^2 - x^2)y$
2	1	-1	$x^2(3y^2 - x^2)$
2	0	0	$y^4 - 4x^2y^2 + x^4$

ℓ	m'	m	$d^{\ell}_{m'm}(z)$
3	3	3	y^6
3	3	2	$-\sqrt{6}\,xy^5$
3	3	1	$\sqrt{15}\,x^2y^4$
3	3	0	$-2\sqrt{5}\,x^3y^3$
3	3	-1	$\sqrt{15}\,x^4y^2$
3	3	-2	$-\sqrt{6}\,x^5y$
3	3	-3	x^6
3	2	2	$(y^2 - 5x^2)y^4$
3	2	1	$-\sqrt{10}\,x(y^2 - 2x^2)y^3$
3	2	0	$\sqrt{30}\,x^2(y^2 - x^2)y^2$
3	2	-1	$-\sqrt{10}\,x^3(2y^2 - x^2)y$
3	2	-2	$x^4(5y^2 - x^2)$
3	1	1	$(y^4 - 8x^2y^2 + 6x^4)y^2$
3	1	0	$-2\sqrt{3}\,x(y^4 - 3x^2y^2 + x^4)y$
3	1	-1	$x^2(6y^4 - 8x^2y^2 + x^4)$
3	0	0	$y^6 - 9x^2y^4 + 9x^4y^2 - x^6$

The d-functions for $\ell = 4$ are given below:

ℓ	m'	m	$d^{\ell}_{m'm}(z)$
4	4	4	y^8
4	4	3	$-2\sqrt{2}\,xy^7$
4	4	2	$2\sqrt{7}\,x^2y^6$
4	4	1	$-2\sqrt{14}\,x^3y^5$
4	4	0	$\sqrt{70}\,x^4y^4$
4	4	-1	$-2\sqrt{14}\,x^5y^3$
4	4	-2	$2\sqrt{7}\,x^6y^2$
4	4	-3	$-2\sqrt{2}\,x^7y$
4	4	-4	x^8

ℓ	m'	m	$d^{\ell}_{m'm}(z)$
4	3	3	$(y^2-7x^2)y^6$
4	3	2	$-\sqrt{14}\,x(y^2-3x^2)y^5$
4	3	1	$\sqrt{7}\,x^2(3y^2-5x^2)y^4$
4	3	0	$-2\sqrt{35}\,x^3(y^2-x^2)y^3$
4	3	-1	$\sqrt{7}\,x^4(5y^2-3x^2)y^2$
4	3	-2	$-\sqrt{14}\,x^5(3y^2-x^2)y$
4	3	-3	$x^6(7y^2-x^2)$
4	2	2	$(y^4-12x^2y^2+15x^4)y^4$
4	2	1	$-\sqrt{2}\,x(3y^4-15x^2y^2+10x^4)y^3$
4	2	0	$\sqrt{10}\,x^2(3y^4-8x^2y^2+3x^4)y^2$
4	2	-1	$-\sqrt{2}\,x^3(10y^4-15x^2y^2+3x^4)y$
4	2	-2	$x^4(15y^4-12x^2y^2+x^4)$
4	1	1	$(y^6-15x^2y^4+30x^4y^2-10x^6)y^2$
4	1	0	$-2\sqrt{5}\,x(y^6-6x^2y^4+6x^4y^2-x^6)y$
4	1	-1	$x^2(10y^6-30x^2y^4+15x^4y^2-x^6)$
4	0	0	$y^8-16x^2y^6+36x^4y^4-16x^6y^2+x^8$

Next an alternative expression for the function $d^{\ell}_{m'm}(\theta)$ is given as a ratio of two polynomials in a single variable. One finds, collecting the terms of exponent n in (94) into a single term,

$$d^{\ell}_{m'm}(\theta) = \left(\cos\frac{\theta}{2}\right)^{2\ell} e^{\ell}_{m'm}\left(\tan\frac{\theta}{2}\right) \tag{99}$$

where the e-function is now a polynomial of order up to 2ℓ in a single variable $u = \tan(\theta/2)$ given by

$$e^{\ell}_{m'm}(u) = (-)^{m'-m}\, u^{m'-m}\, [(\ell+m')!(\ell-m')!(\ell+m)!(\ell-m)!]^{1/2} \times \sum_{k=k_1}^{k_2} \frac{(-)^k\, u^{2k}}{(\ell-m'-k)!(\ell+m-k)!(m'-m+k)!k!} \tag{100}$$

One may re-express the d-functions in terms of the variable u only, as follows:

$$d^{\ell}_{m'm}(u) = \frac{e^{\ell}_{m'm}(u)}{(1+u^2)^{\ell}} \tag{101}$$

It is seen that this function remains finite, when $u \to \infty$ as $\theta \to \pi$.

It is clear that all the symmetry relations for the d-functions apply to the e-functions as well. Thus, one has

$$\begin{aligned} e^{\ell}_{m'm}(-u) &= (-)^{m'-m}\, e^{\ell}_{m'm}(u) \\ e^{\ell}_{m'm}(u) &= (-)^{m'-m}\, e^{\ell}_{mm'}(u) \\ e^{\ell}_{m'm}(u) &= (-)^{m'-m}\, e^{\ell}_{-m'-m}(u) \\ e^{\ell}_{m'm}(1/u) &= (-)^{\ell-m'}\, e^{\ell}_{m'-m}(u) \end{aligned} \tag{102}$$

The e-functions are tabulated here for ℓ up to 3:

ℓ	m'	m	$e^{\ell}_{m'm}(u)$
0	0	0	1
1	1	1	1
1	1	0	$-\sqrt{2}\,u$
1	1	-1	u^2
1	0	0	$1-u^2$
2	2	2	1
2	2	1	$-2u$
2	2	0	$\sqrt{6}\,u^2$
2	2	-1	$-2u^3$
2	2	-2	u^4
2	1	1	$1-3u^2$
2	1	0	$-\sqrt{6}\,u(1-u^2)$
2	1	-1	$u^2(3-u^2)$
2	0	0	$1-4u^2+u^4$

ℓ	m'	m	$e^{\ell}_{m'm}(u)$
3	3	3	1
3	3	2	$-\sqrt{6}\,u$
3	3	1	$\sqrt{15}\,u^2$
3	3	0	$-2\sqrt{5}\,u^3$
3	3	-1	$\sqrt{15}\,u^4$
3	3	-2	$-\sqrt{6}\,u^5$
3	3	-3	u^6
3	2	2	$1-5u^2$
3	2	1	$-\sqrt{10}\,u(1-2u^2)$
3	2	0	$\sqrt{30}\,u^2(1-u^2)$
3	2	-1	$-\sqrt{10}\,u^3(2-u^2)$
3	2	-2	$u^4(5-u^2)$
3	1	1	$1-8u^2+6u^4$
3	1	0	$-2\sqrt{3}\,u(1-3u^2+u^4)$
3	1	-1	$u^2(6-8u^2+u^4)$
3	0	0	$1-9u^2+9u^4-u^6$

The e-functions for $\ell = 4$ are given below:

ℓ	m'	m	$e^{\ell}_{m'm}(u)$
4	4	4	1
4	4	3	$-2\sqrt{2}\,u$
4	4	2	$2\sqrt{7}\,u^2$
4	4	1	$-2\sqrt{14}\,u^3$
4	4	0	$\sqrt{70}\,u^4$
4	4	-1	$-2\sqrt{14}\,u^5$
4	4	-2	$2\sqrt{7}\,u^6$
4	4	-3	$-2\sqrt{2}\,u^7$
4	4	-4	u^8

ℓ	m'	m	$e^{\ell}_{m'm}(u)$
4	3	3	$1-7u^2$
4	3	2	$-\sqrt{14}\,u(1-3u^2)$
4	3	1	$\sqrt{7}\,u^2(3-5u^2)$
4	3	0	$-2\sqrt{35}\,u^3(1-u^2)$
4	3	-1	$\sqrt{7}\,u^4(5-3u^2)$
4	3	-2	$-\sqrt{14}\,u^5(3-u^2)$
4	3	-3	$u^6(7-u^2)$
4	2	2	$1-12u^2+15u^4$
4	2	1	$-\sqrt{2}\,u(3-15u^2+10u^4)$
4	2	0	$\sqrt{10}\,u^2(3-8u^2+3u^4)$
4	2	-1	$-\sqrt{2}\,u^3(10-15u^2+3u^4)$
4	2	-2	$u^4(15-12u^2+u^4)$
4	1	1	$1-15u^2+30u^4-10u^6$
4	1	0	$-2\sqrt{5}\,u(1-6u^2+6u^4-u^6)$
4	1	-1	$u^2(10-30u^2+15u^4-u^6)$
4	0	0	$1-16u^2+36u^4-16u^6+u^8$

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