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Treatment of Particles with Spin in the Final State: Sequential Decays involving $\omega \to \gamma + \pi^0$ and $N\pi\pi$ Systems

—Version III—

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abstract

If the decay $\omega \to \gamma + \pi^0$ is involved in parallel sequential decays, then it is essential that a single helicity frame be used for the ω decay. The same comments apply to an analysis involving the treatment of N in $N\pi\pi$ systems.

It is shown that the decay amplitudes in canonical formalism provide an efficient method for dealing with non-zero spins in the final states.

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1 Introduction

Consider a three-body system consisting of $(s + \pi_1 + \pi_1')$, with two possible intermediate states $j \to s + \pi_1$ and $j' \to s + \pi_1'$, which is followed by $s \to s_1 + \pi_2$. We will take a concrete example where s is the ω, with a decay chain $\omega \to \gamma + \pi^0$. In this case then, s_1 is a photon, and so $s = s_1 = 1$ and $\pi_2 = \pi^0$.

Let J be the spin of the parent system. Then we have

$$\begin{aligned} &J \to j(\Omega_0) + \pi_1', && j \to s(\Omega) + \pi_1, && s \to s_1(\Omega_2) + \pi_2 \\ &J \to j'(\Omega_0') + \pi_1, && j' \to s(\Omega') + \pi_1', && s \to s_1(\Omega_2') + \pi_2 \end{aligned} \tag{1.1}$$

where $\Omega_0 = (\theta_0, \phi_0)$ is the direction of j in the parent rest frame, and similarly for j'; Ω describes s in the j RF (rest frame), while Ω_2 refers to s_1 in the s RF.

The decay $\omega \to \gamma + \pi^0$ must be described by a single frame in a given problem, but there are, in our example (1.1), three different frames Ω_2 and Ω_2' in which the decay amplitudes are given. So we need to recast them into a single given frame. The purpose of this note is to show how this can be accomplished and illustrated with a simple but important reaction.

We shall employ the helicity formalism to describe the 'parallel sequential decays' given in (1.1). The canonical and helicity rest frames are illustrated in Fig.1*b*.

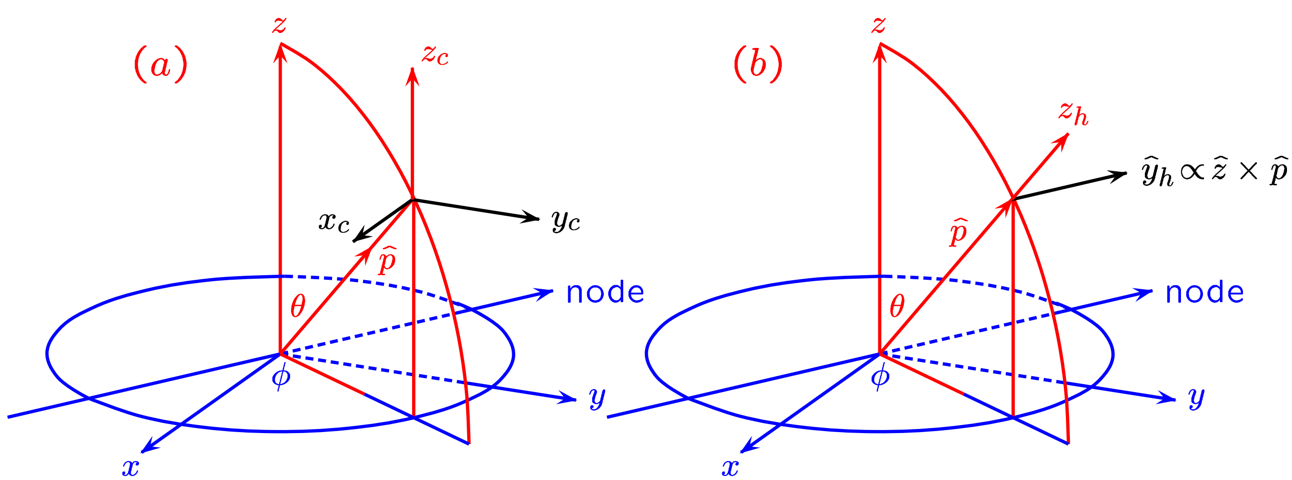

Figure 1: The orientation of the coordinate systems associated with a particle at rest in the (a) canonical ($\hat{x}_c$, $\hat{y}_c$, $\hat{z}_c$), and (b) helicity description ($\hat{x}_h = \hat{y}_h \times \hat{z}_h$, $\hat{y}_h \propto \hat{z} \times \hat{p}$, $\hat{z}_h = \hat{p}$).

In Section 2, we consider the decay amplitudes for j, j' and s to illustrate the principles; in section 3 we treat the decay of J as well—for a simple, but practically important, example. Section 4 is reserved for a treatment of N in the $N\pi\pi$ system. The decay amplitudes in canonical formalism are given in Section 5. Conclusions are given Section 6.

2 Parallel Sequential Decays

We use the helicity description for the decay amplitude for $j \to s + \pi_1$

$$A^j_{\lambda_j\,\lambda}(\Omega) \;= N_j\, F^j_\lambda\, D^{j\,*}_{\lambda_j\lambda}(\phi,\theta,0), \qquad N_j = \sqrt{\frac{2j+1}{4\pi}} \tag{2.1a}$$

$$F^j_\lambda \;= \sum_\ell \left(\frac{2\ell+1}{2j+1}\right)^{1/2} G^j_\ell\,(\ell 0\, s\lambda|j\lambda) \tag{2.1b}$$

where $\Omega = (\theta, \phi)$ describes the direction of s in the j RF (rest frame) [see Fig. 1b], and G^j_ℓ is the decay coupling constant for $j \to s + \pi_1$ with an orbital angular momentum ℓ. The decay amplitude for $j \to s + \pi_1$, followed by $s \to s_1 + \pi_2$, is

$$\begin{aligned} A^j_{\lambda_j\,\lambda_1}(\Omega,\Omega_2) \;&= N_j\, N_s \sum_\lambda\; A^j_{\lambda_j\,\lambda}(\Omega)\; f^{\,s}_{\lambda_1}\, D^{s\,*}_{\lambda\,\lambda_1}(\phi_2,\theta_2,0), \qquad N_s = \sqrt{\frac{2s+1}{4\pi}} \\ &= N_j\, N_s \sum_\lambda\; F^j_\lambda\, D^{j\,*}_{\lambda_j\lambda}(\phi,\theta,0)\; f^{\,s}_{\lambda_1}\; D^{s\,*}_{\lambda\,\lambda_1}(\phi_2,\theta_2,0) \end{aligned} \tag{2.2a}$$

$f^{\,s}_{\lambda_1}$ is the helicity-coupling amplitude corresponding to $s \to s_1 + \pi_2$. For the example of $\omega \to \gamma + \pi^0$, we have $f^{\,s}_\pm = -f^{\,s}_\mp$ and $f^{\,s}_0 = 0$. The angles $\Omega_2 = (\theta_2, \phi_2)$ describes the direction of s_1 in the s RF [see Fig. 1b].

The amplitude corresponding to the decay chain $j' \to s + \pi_1'$, followed by $s \to s_1 + \pi_2$, is

$$A^{j'}_{\lambda_j'\,\lambda_1}(\Omega',\Omega_2') \;= N_{j'}\, N_s \sum_\lambda\; F^{j'}_\lambda\, D^{j'\,*}_{\lambda_j'\lambda}(\phi',\theta',0)\; f^{\,s}_{\lambda_1}\; D^{s\,*}_{\lambda\,\lambda_1}(\phi_2',\theta_2',0), \qquad N_{j'} = \sqrt{\frac{2j'+1}{4\pi}}$$

$$F^{j'}_\lambda \;= \sum_{\ell'} \left(\frac{2\ell'+1}{2j'+1}\right)^{1/2} G^{j'}_{\ell'}\,(\ell' 0\, s\lambda|j'\lambda) \tag{2.2b}$$

The angles $\Omega' = (\theta', \phi')$ correspond to the direction s in the j' RF, while the angles $\Omega_2' = (\theta_1', \phi_2')$ describe the direction of s_1 in the s RF. It is clear that the angles Ω_2 and Ω_2' are different, because of the different paths taken to get to the s RF.

We need to employ a single amplitude for the decay $s \to s_1 + \pi_2$. For this purpose, we note that there is yet another way to describe the s decay; we can in fact go directly from the J RF to the s RF, without going through the intermediate steps of j, and j'. The decay amplitude for this case is

$$A^s_{\lambda\,\lambda_1}(\Omega_1) = N_s\, f^{\,s}_{\lambda_1}\, D^{s\,*}_{\lambda\,\lambda_1}(\phi_1,\theta_1,0) \tag{2.3}$$

The angles $\Omega_1 = (\theta_1, \phi_1)$ are of course different from Ω_2 and Ω_2'.

3 Amplitudes for $\bar{p}p \to \omega + \pi^0 + \pi^0$

It is instructive to apply the above results to a problem considered by Giarritta[2]:

$$\bar{p}p|_{\rm rest}(^3S_1 \text{ or } ^1P_1) \to \omega + \pi^0(\pi_1) + \pi^0(\pi_1') \qquad \omega \to \gamma + \pi^0(\pi_2) \tag{3.1}$$

So we have $J = 1$ for the parent system. We fix the coordinate system in the decay plane, such that the z-axis is along the direction of ω and the y-axis is along the decay normal, $\hat{y} \propto \vec{\omega} \times \vec{\pi}_1$. We shall consider two intermediate states $b_1(1235) \to \omega + \pi_1$ and $b_1'(1235) \to \omega + \pi_1'$, so that we have $j = j' = s = s_1 = 1$. The analogue of (1.1) for this example is

$$\begin{aligned} &J \xrightarrow[L]{} j(\Omega_0) + \pi_1', \qquad && j \xrightarrow[\ell]{} s(\Omega) + \pi_1, \qquad && s \to s_1(\Omega_2) + \pi_2 \\ &J \xrightarrow[L']{} j'(\Omega_0') + \pi_1, \qquad && j' \xrightarrow[\ell']{} s(\Omega') + \pi_1', \qquad && s \to s_1(\Omega_2') + \pi_2 \end{aligned} \tag{3.2}$$

where L (L') and ℓ (ℓ') are the orbital angular momenta in J and $j(j')$ RFs, respectively.

The processes outlined in (3.2) are illustrated in Fig. 2. The amplitudes for $J \to j(\Omega_0)+\pi_1'$

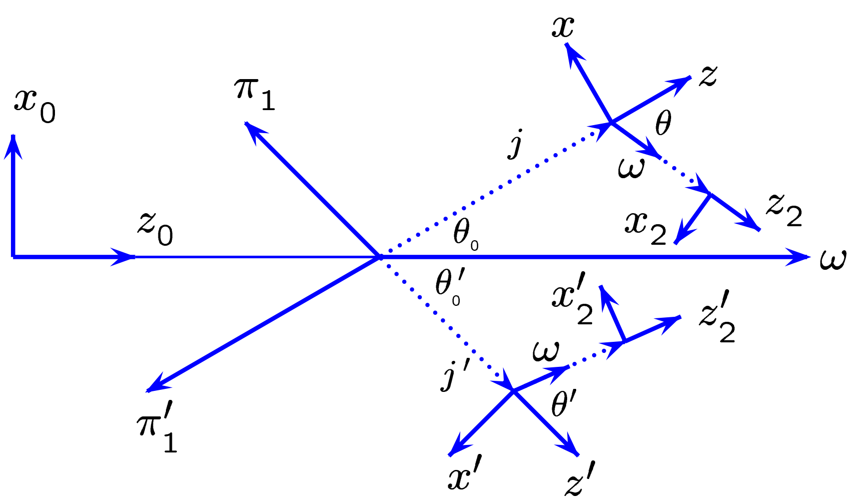

Figure 2: The process $\bar{p}p \to \omega + \pi_1 + \pi_1'$. See (3.2) for the notations. From the $\bar{p}p$ RF(rest frame), we go into the j RF or j' RF and then to the ω RF via pure time-like Lorentz transformations. The coordinate system $(\hat{x}_0, \hat{y}_0, \hat{z}_0)$ in the $\bar{p}p$ RF is such that $\hat{z}_0$ is along the direction of ω and $\hat{y}_0$ is along the normal to the reaction plane (out of the paper). The helicity frames in j RF and j' RF are denoted $(\hat{x}, \hat{y}, \hat{z})$ and $(\hat{x}', \hat{y}', \hat{z}')$. Two helicity frames for the ω RF are shown: $(\hat{x}_2, \hat{y}_2, \hat{z}_2)$ and $(\hat{x}_2', \hat{y}_2', \hat{z}_2')$. The relevant angles are $\Omega_0 = (\theta_0, 0)$, $\Omega_0' = (\theta_0', \pi)$, $\Omega = (\theta, -\pi)$ and $\Omega' = (\theta', -\pi)$. The third helicity frame for the ω RF coincides with $(\hat{x}_0, \hat{y}_0, \hat{z}_0)$.

and $J \to j'(\Omega_0') + \pi_1$ are

$$\begin{aligned} A^{J\,j}_{M\,\lambda_j}(\Omega_0) &= N_J\, H^J_{\lambda_j}\, D^{J\,*}_{M\,\lambda_j}(0,\theta_0,0) = H^J_{\lambda_j}\, d^{\,J}_{M\,\lambda_j}(\theta_0), \qquad N_J = \sqrt{\frac{2J+1}{4\pi}} \\ A^{J\,j'}_{M\,\lambda_j'}(\Omega_0') &= N_J\, \bar{H}^J_{\lambda_j'}\, D^{J\,*}_{M\,\lambda_j'}(\pi,\theta_0',0) = \exp[i\,M\,\pi]\, \bar{H}^J_{\lambda_j'}\, d^{\,J}_{M\,\lambda_j'}(\theta_0') \end{aligned} \tag{3.3}$$

where

$$
\begin{aligned}
H^J_{\lambda_j} &= \sum_L \left(\frac{2L+1}{2J+1}\right)^{1/2} K^J_L\,(L0\,j\lambda_j|J\lambda_j) \\
\bar{H}^J_{\lambda'_j} &= \sum_{L'} \left(\frac{2L+1}{2J+1}\right)^{1/2} \bar{K}^J_{L'}\,(L'0\,j'\lambda'_j|J\lambda'_j)
\end{aligned}
\tag{3.4}
$$

We allow for different states for j and j'; for example, the j could stand for the $b_1(1235)$, while the j' might represent the $\rho(1700)$. The 'bar's over H^J and K^J indicate different amplitudes for these states. However, if both intermediate states happen to be the $b_1(1235)$, then the Bose symmetrization requires that $\bar{H}^J = H^J$ and $\bar{K}^J = K^J$.

A state $|s\lambda\rangle$ for ω in $(\hat{x}_2, \hat{y}_2, \hat{z}_2)$, more precisely to be denoted $|s\lambda\rangle_2$, can be described by a state $|s\nu\rangle_0$ given in $(\hat{x}_0, \hat{y}_0, \hat{z}_0)$ via

$$
|s\lambda\rangle_0 = R(\pi,\beta,0)\,|s\lambda\rangle_2 \tag{3.5}
$$

so that

$$
|s\lambda\rangle_2 = R^\dagger(\pi,\beta,0)\,|s\lambda\rangle_0 = \sum_\nu |s\nu\rangle_0\;{}_0\langle s\nu|\,R^\dagger(\pi,\beta,0)\,|s\lambda\rangle_0 = \sum_\nu D^{s\,*}_{\lambda\,\nu}(\pi,\beta,0)\,|s\nu\rangle_0 \tag{3.6}
$$

using unitarity of the D-functions. The decay amplitude (2.1) must be modified [see Appendix A] according to

$$
\begin{aligned}
A^j_{\lambda_j\,\lambda}(\Omega) \equiv {}_2\langle s\lambda|\psi\rangle \quad \Longrightarrow \quad {}_0\langle s\nu|\psi\rangle &= \sum_\lambda {}_0\langle s\nu|s\lambda\rangle_2\;{}_2\langle s\lambda|\psi\rangle \\
&= \sum_\lambda D^{s\,*}_{\lambda\,\nu}(\pi,\beta,0)\;{}_2\langle s\lambda|\psi\rangle \equiv A^j_{\lambda_j\,\nu}(\Omega,R)
\end{aligned}
\tag{3.7}
$$

so that it is now measured with respect to the state $|s\nu\rangle_0$ with a sum over λ

$$
\begin{aligned}
A^j_{\lambda_j\,\nu}(\Omega,R) &= \sum_\lambda A^j_{\lambda_j\,\lambda}(\Omega)\,D^{s\,*}_{\lambda\,\nu}(\pi,\beta,0) \\
&= N_j \sum_\lambda F^j_\lambda\,D^{j\,*}_{\lambda_j\lambda}(-\pi,\theta,0)\,D^{s\,*}_{\lambda\,\nu}(\pi,\beta,0) \\
&= N_j \sum_\lambda (-)^{-\lambda_j+\lambda}\,F^j_\lambda\,d^{\,j}_{\lambda_j\lambda}(\theta)\,d^{\,s}_{\lambda\,\nu}(\beta)
\end{aligned}
\tag{3.8}
$$

where $R = R(\pi,\beta,0)$ and $\Omega = (\theta,-\pi)$. So the overall decay amplitude is

$$
A^{J\,j}_{M\,\nu}(\Omega_0,\Omega,R) = N_J\,N_j \sum_{\lambda_j} H^J_{\lambda_j}\,d^{\,J}_{M\,\lambda_j}(\theta_0) \sum_\lambda (-)^{\lambda_j-\lambda}\,F^j_\lambda\,d^{\,j}_{\lambda_j\lambda}(\theta)\,d^{\,s}_{\lambda\,\nu}(\beta) \tag{3.9}
$$

Note that the appearance of two rotations $R(\pi,\beta,0)$ and $R(-\pi,\theta,0)$ with the second rotation around the z-axis by $-\pi$. Consider a special case with $\theta_0 = \theta = \beta = 0$. For this case, we need to ensure that there be no net rotation of the coordinate axes, because there would have been a rotation around by 2π and a spurious phase $(-)^{2\,m\,\pi}$ for a z-component of spin m, had the second rotation been $R(+\pi,\theta,0)$ instead of $R(-\pi,\theta,0)$.

Consider next the decay amplitude for $J \to j' + \pi - 1$ with $j' \to s + \pi_1'$. The $|s\lambda\rangle_2'$ for ω in $(\hat{x}_2', \hat{y}_2', \hat{z}_2')$ can be expressed by

$$\begin{aligned} |s\lambda\rangle_0 &= R(0,\beta',0)\,|s\lambda\rangle_2' \\ |s\lambda\rangle_2' &= R^\dagger(0,\beta',0)\,|s\lambda\rangle_0 = \sum_\nu d^s_{\lambda\,\nu}(\beta')\,|s\nu\rangle_0 \end{aligned} \tag{3.10}$$

The overall decay amplitude with respect to the state $|s\nu\rangle_0$ is, with $R' = R(0,\beta',0)$ and $\Omega' = (\theta', -\pi)$,

$$A^{J\,j'}_{M\,\nu}(\Omega_0, \Omega', R') = N_J\, N_{j'} \sum_{\lambda_j'} (-)^{M-\lambda_j'}\, \bar{H}^J_{\lambda_j'}\, d^J_{M\,\lambda_j'}(\theta_0') \sum_\lambda F^{j'}_\lambda\, d^{j'}_{\lambda_j'\lambda}(\theta')\, d^s_{\lambda\,\nu}(\beta') \tag{3.11}$$

Once again, we note that there are two rotations $R(\pi, \theta_0', 0)$ and $R(-\pi, \theta', 0)$, with the second z-rotation given by $-\pi$. The $|s\nu\rangle_0$ decay itself, i.e. $\omega \to \gamma + \pi^0$, is given in the standard helicity prescription

$$\langle \Omega_1, \lambda_1 | \mathcal{M}_s | s\nu\rangle_0 = N_s\, f^s_{\lambda_1}\, D^{s\,*}_{\nu\,\lambda_1}(\phi_1, \theta_1, 0) \tag{3.12}$$

So we find, summing over ν,

$$\begin{aligned} A^{J\,j}_{M\,\lambda_1}(\Omega_0, \Omega, R, \Omega_1) &= N_J\, N_j \sum_{\lambda_j} H^J_{\lambda_j}\, d^J_{M\,\lambda_j}(\theta_0) \\ &\times \sum_\lambda (-)^{\lambda_j-\lambda}\, F^j_\lambda\, d^j_{\lambda_j\lambda}(\theta) \sum_\nu d^s_{\lambda\,\nu}(\beta)\; f^s_{\lambda_1}\, D^{s\,*}_{\nu\,\lambda_1}(\phi_1, \theta_1, 0) \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} A^{J\,j'}_{M\,\lambda_1}(\Omega_0', \Omega', R', \Omega_1) &= N_J\, N_{j'} \sum_{\lambda_j'} (-)^{M-\lambda_j'}\, \bar{H}^J_{\lambda_j'}\, d^J_{M\,\lambda_j'}(\theta_0') \\ &\times \sum_\lambda F^{j'}_\lambda\, d^{j'}_{\lambda_j'\lambda}(\theta') \sum_\nu d^s_{\lambda\,\nu}(\beta')\; f^s_{\lambda_1}\, D^{s\,*}_{\nu\,\lambda_1}(\phi_1, \theta_1, 0) \end{aligned} \tag{3.14}$$

The formulas above give the s (or γ) and its helicity λ_1 in a single given frame—the desired result and the purpose of this note.

For completeness, we shall work out the third type of isobar for (3.1), i.e. that of the dipion system $\pi_1 + \pi_1'$ described by $|\ell_3\,\lambda_3\rangle$.

$$J \xrightarrow[L_3]{} \ell_3(\Omega_3) + s, \quad s \to s_1(\Omega_1) + \pi_2 \tag{3.15}$$

The overall decay amplitude for $|JM\rangle \to |s\lambda\rangle + |\ell_3\,\lambda_3\rangle$ is

$$\begin{aligned} A^{J\,\ell_3}_{M\,\lambda_1}(\Omega_3, \Omega_1) &= N_J\, N_{\ell_3} \sum_{\lambda\,\lambda_3} E^J_{\lambda\,\lambda_3}\, D^{J\,*}_{M\,\lambda-\lambda_3}(0,0,0) \\ &\times D^{\ell_3\,*}_{\lambda_3\,0}(\phi_3, \theta_3, 0)\, A^s_{\lambda\,\lambda_1}(\Omega_1), \qquad N_{\ell_3} = \sqrt{\frac{2\ell_3+1}{4\pi}} \end{aligned} \tag{3.16}$$

Note that $M = \lambda - \lambda_3$. $E^J_{\lambda\,\lambda_3}$ is the usual helicity-coupling amplitude

$$E^J_{\lambda\,\lambda_3} = \sum_{L_3\,S} \left(\frac{2L_3+1}{2J+1}\right)^{1/2} Q^J_{L_3\,S}\,(L_3\,0\,SM|JM)(s\,\lambda\,\ell_3\; -\lambda_3|SM) \tag{3.17}$$

where

$$|\ell_3 - s| \le S \le \ell_3 + s \quad \text{and} \quad |J-S| \le \ell_0 \le J+S \tag{3.18}$$

$\Omega_3(\theta_3,\phi_3)$ is measured in the dipion RF defined by $(-\hat{x}_0, \hat{y}_0, -\hat{z}_0)$. In general, $\phi_3 = 0$ or $\phi_3 = \pi$, but we can always set $\phi_3 = 0$ by allowing negative values of θ_3, i.e. $-\pi < \theta_3 < \pi$. The overall amplitude becomes

$$A^{J\,\ell_3}_{M\,\lambda_1}(\Omega_3,\Omega_1) = N_J\,N_{\ell_3}\,N_s \sum_{\lambda_3} E^J_{\lambda\,\lambda_3}\,d^{\ell_3}_{\lambda_3\,0}(\theta_3)\;f^{\,s}_{\lambda_1}\,D^{s\,*}_{\lambda\,\lambda_1}(\phi_1,\theta_1,0) \tag{3.19}$$

where $\lambda = M + \lambda_3$ and so there is no summation on λ.

In order to gain insight to the problem at hand, we shall work out the full amplitude incorporating three different isobars. Observe

$$A^J_{M\,\lambda_1} = V_{Jj}\,A^{J\,j}_{M\,\lambda_1}(\Omega_0,\Omega,R,\Omega_1) + V_{Jj'}\,A^{J\,j'}_{M\,\lambda_1}(\Omega'_0,\Omega',R',\Omega_1) + V_{J\ell_3}\,A^{J\,\ell_3}_{M\,\lambda_1}(\Omega_1,\Omega_3) \tag{3.20}$$

where V_{Jj}, $V_{Jj'}$ and $V_{J\ell_3}$ are the parameters (complex in general) which govern the strength of each isobar. The parameters should be a function of J but *not* of either M or the photon helicity λ_1. We see that, absorbing the normalization constants N into V,

$$\begin{aligned}
A^J_{M\,\lambda_1} = V_{Jj} \Bigg\{ & \sum_{\lambda_j} H^J_{\lambda_j}\,d^J_{M\,\lambda_j}(\theta_0) \sum_{\lambda} (-)^{\lambda_j-\lambda}\,F^j_\lambda\,d^j_{\lambda_j\lambda}(\theta) \\
& \times \sum_\nu d^s_{\lambda\,\nu}(\beta) \Bigg\} f^{\,s}_{\lambda_1}\,D^{s\,*}_{\nu\,\lambda_1}(\phi_1,\theta_1,0) \\
+V_{Jj'} \Bigg\{ & \sum_{\lambda'_j} (-)^{M-\lambda'_j}\,\bar{H}^J_{\lambda'_j}\,d^J_{M\,\lambda'_j}(\theta'_0) \sum_\lambda F^{j'}_\lambda\,d^{j'}_{\lambda'_j\lambda}(\theta') \\
& \times \sum_\nu d^s_{\lambda\,\nu}(\beta') \Bigg\} f^{\,s}_{\lambda_1}\,D^{s\,*}_{\nu\,\lambda_1}(\phi_1,\theta_1,0) \\
+V_{J\ell_3} \Bigg\{ & \sum_{\lambda_3} E^J_{\nu\,\lambda_3}\,d^{\ell_3}_{\lambda_3\,0}(\theta_3) \Bigg\} f^{\,s}_{\lambda_1}\,D^{s\,*}_{\nu\,\lambda_1}(\phi_1,\theta_1,0)
\end{aligned} \tag{3.21}$$

where $\nu = M + \lambda_3$ in the third term. The decay amplitude for $\omega \to \gamma + \pi^0$ can now be factored out in the expression given above.

4 $N\pi\pi$ Systems

Consider the system $N\pi_1\pi_1'$ where N is a nucleon and there are two possible isobars $N\pi_1$ and $N\pi_1'$. We use Fig. 2 in which the ω replaced by a nucleon N. So we now have $s = 1/2$. The appropriate decay amplitude for a final state containing $|s\nu\rangle_0$ have already been given in (3.9) and (3.11). The full amplitude is, absorbing the normalization constants N into V,

$$\begin{aligned} A^J_{M\,\nu} = V_{Jj} & \left\{ \sum_{\lambda_j} H^J_{\lambda_j}\, d^J_{M\,\lambda_j}(\theta_0) \sum_{\lambda} (-)^{\lambda_j - \lambda}\, F^j_\lambda\, d^j_{\lambda_j \lambda}(\theta)\, d^s_{\lambda\,\nu}(\beta) \right\} \\ + V_{Jj'} & \left\{ \sum_{\lambda_j'} (-)^{M-\lambda_j'}\, \bar{H}^J_{\lambda_j'}\, d^J_{M\,\lambda_j'}(\theta_0') \sum_{\lambda} F^{j'}_\lambda\, d^{j'}_{\lambda_j' \lambda}(\theta')\, d^s_{\lambda\,\nu}(\beta'\,) \right\} \\ + V_{J\ell_3} & \, E^J_{\nu\,\lambda_3}\, d^{\ell_3}_{\lambda_3\,0}(\theta_3) \end{aligned} \tag{4.1}$$

where $\nu = M + \lambda_3$. All three amplitudes above are now expressed in terms of a single nucleon state $|s\nu\rangle_0$ defined in the coordinate system $(\hat{x}_0, \hat{y}_0, \hat{z}_0)$.

5 Alternative Approach

The extra rotations by the Euler angles of previous sections can be avoided if canonical frames had been employed for the intermediate states [see Fig. 1a]. See Appendix B for a canonical prescription for dealing with general two-body decays.

For $j \to s + \pi_1$ we have

$$\begin{aligned} A^j_{m_j \nu}(\Omega_c) & \propto \langle \Omega_c\, s\nu | \mathcal{M}_j | j m_j \rangle \\ & \propto \sum_\ell \langle \Omega_c\, s\nu | j m_j\, \ell\,\rangle \langle j m_j\, \ell | \mathcal{M}_j | j m_j \rangle \end{aligned} \tag{5.1}$$

where Ω_c describes the the direction of s in the canonical jRF and ν is the z-component of spin s in the canonical sRF. Setting

$$G^j_\ell \propto \langle j m_j\, \ell | \mathcal{M}_j | j m_j \rangle \tag{5.2}$$

we see that, with $m = m_j - \nu$,

$$\begin{aligned} A^j_{m_j \nu}(\Omega_c) & = \sum_\ell G^j_\ell\, (\ell\, m\, s\nu | j m_j)\, Y^\ell_m(\Omega_c) \\ & = \sum_\ell N_\ell\, G^j_\ell\, (\ell\, m\, s\nu | j m_j)\, D^{\ell\,*}_{m\,0}(\phi_c, \theta_c, 0) \\ & = \sum_\ell N_\ell\, G^j_\ell\, (\ell\, m\, s\nu | j m_j)\, d^\ell_{m\,0}(-\theta_c), \quad \theta_c > 0 \\ & = (-)^{m_j - \nu} \sum_\ell N_\ell\, G^j_\ell\, (\ell\, m\, s\nu | j m_j)\, d^\ell_{m\,0}(\theta_c), \quad \theta_c > 0 \end{aligned} \tag{5.3}$$

where $\phi_c = 0$ and, from Appendix A of Ref.[1],

$$Y^{\ell}_{m}(\Omega) = N_\ell \, D^{\ell\,*}_{m\,0}(\phi, \theta, 0), \qquad N_\ell = \sqrt{\frac{2\ell+1}{4\pi}} \tag{5.4}$$

Likewise, for $J \to j + \pi_1'$ we find

$$\begin{aligned} A^{J\,j}_{M\,m_j}(\Omega_0) &= \sum_L N_L \, K^J_L \, (LM_{_L}\, jm_j|JM) \, D^{L\,*}_{M_L 0}(\phi_0, \theta_0, 0), \qquad N_L = \sqrt{\frac{2L+1}{4\pi}} \\ &= \sum_L N_L \, K^J_L \, (LM_{_L}\, jm_j|JM) \, d^L_{M_L 0}(\theta_0) \end{aligned} \tag{5.5}$$

with $\phi_0 = 0$ and $M_{_L} = M - m_j$.

The decay amplitude for $J \to j + \pi_1'$ followed by $j \to s + \pi_1$ is

$$\begin{aligned} A^{J\,j}_{M\,\nu}(\Omega_0, \Omega_c) = \sum_{L\,m_j} & N_L \, K^J_L \, (LM_{_L}\, jm_j|JM) \, d^L_{M_L 0}(\theta_0) \\ & \times (-)^{m_j-\nu} \sum_\ell N_\ell \, G^j_\ell \, (\ell\, m\, s\nu|jm_j) \, d^\ell_{m\,0}(\theta_c) \end{aligned} \tag{5.6}$$

Consider a special case $L = M_{_L} = \ell = m = 0$. We then see that $J = j = s$ and $M = m_j = \nu$. In this case, the amplitude is *independent* of the angles Ω_0 and Ω_c and it is proportional to

$$A^{J\,j}_{M\,\nu}(\Omega_0, \Omega_c) = \frac{1}{4\pi} \, K^J_0 \, G^j_0 \tag{5.7}$$

where $M = m_j = \nu$. But we must obtain the same result from (3.9). For the purpose, we first note that

$$H^J_{\lambda_j} = \frac{1}{\sqrt{2J+1}} K^J_0 \quad \text{and} \quad F^j_\lambda = \frac{1}{\sqrt{2j+1}} G^j_0 \tag{5.8}$$

independent of the helicities. Setting J and s to j, and using the well-known property of the d-functions [3]

$$d^j_{m'm}(-\beta) = (-)^{m'-m} \, d^j_{m'm}(\beta), \quad d^j_{mm'}(\beta) = (-)^{m'-m} \, d^j_{m'm}(\beta), \tag{5.9}$$

we find

$$\begin{aligned} A^{J\,j}_{M\,\nu}(\Omega_0, \Omega, R) &= N_j^2 \sum_{\lambda_j} H^j_{\lambda_j} \, d^j_{M\,\lambda_j}(\theta_0) \sum_\lambda (-)^{\lambda_j - \lambda} \, F^j_\lambda \, d^j_{\lambda_j \lambda}(\theta) \, d^j_{\lambda\,\nu}(\beta) \\ &= N_j^2 \sum_{\lambda_j} (-)^{M-\lambda_j} \, H^j_{\lambda_j} \, d^j_{M\,\lambda_j}(-\theta_0) \sum_\lambda F^j_\lambda \, d^j_{\lambda_j \lambda}(\theta) \, d^j_{\lambda\,\nu}(-\beta) \\ &= \frac{(-)^{M-\nu}}{4\pi} \, K^J_0 \, G^j_0 \sum_{\lambda_j} d^j_{M\,\lambda_j}(-\theta_0) \sum_\lambda d^j_{\lambda_j \lambda}(\theta) \, d^j_{\lambda\,\nu}(-\beta) \\ &= \frac{1}{4\pi} \, K^J_0 \, G^j_0 \, d^j_{M\,\nu}(-\theta_0 + \theta - \beta) = \frac{1}{4\pi} \, K^J_0 \, G^j_0 \end{aligned} \tag{5.10}$$

since $M = \nu$ and $-\theta_0 + \theta - \beta = 0$. That θ is equal to $\theta_0 + \beta$ can be seen in Fig. 2, by drawing the axes z_0 and x_0 at jRF, but this can also be checked explicitly by working out one example with relevant angles in detail (see Appendix C).

Likewise, the decay amplitude for $J \to j' + \pi_1$ followed by $j' \to s + \pi_1'$ is

$$\begin{aligned} A^{J\,j'}_{M\,\nu}(\Omega_0', \Omega_c') = &\sum_{L'\,m_j'} N_{L'}\, K^J_{L'}\, (L'M_L'\, j'm_j'|JM)\; d^{L'}_{M_L'\,0}(\theta_0') \\ &\times (-)^{m_j'-\nu} \sum_{\ell\,\nu} N_\ell\, G^{j'}_\ell\, (\ell\, m\, s\nu|j'm_j')\; d^\ell_{m\,0}(\theta_c') \end{aligned} \tag{5.11}$$

Again, if $L' = M_L' = \ell = m = 0$, then the amplitude is *independent* of the angles Ω_0' and Ω_c' and it is given by

$$A^{J\,j'}_{M\,\nu}(\Omega_0', \Omega_c') = \frac{1}{4\pi}\, K^J_0\, G^{j'}_0 \tag{5.12}$$

where $M = m_j' = \nu$. We should obtain the same result from (3.11). Setting $J = j' = s$, we obtain

$$\begin{aligned} A^{J\,j'}_{M\,\nu}(\Omega_0, \Omega', R') &= N_J\, N_{j'} \sum_{\lambda_j'} (-)^{M-\lambda_j'}\, \bar{H}^J_{\lambda_j'}\, d^J_{M\,\lambda_j'}(\theta_0') \sum_\lambda F^{j'}_\lambda\, d^{j'}_{\lambda_j'\lambda}(\theta')\, d^s_{\lambda\,\nu}(\beta') \\ &= N^2_{j'} \sum_{\lambda_j'} \bar{H}^J_{\lambda_j'}\, d^J_{M\,\lambda_j'}(-\theta_0') \sum_\lambda F^{j'}_\lambda\, d^{j'}_{\lambda_j'\lambda}(\theta')\, d^s_{\lambda\,\nu}(\beta') \\ &= \frac{1}{4\pi}\, K^J_0\, G^{j'}_0\, d^{j'}_{M\,\nu}(-\theta_0' + \theta' + \beta') = \frac{1}{4\pi}\, K^J_0\, G^{j'}_0 \end{aligned} \tag{5.13}$$

since $M = \nu$ and $-\theta_0' + \theta + \beta' = 0$ (see Appendix C).

Because of the use of canonical rest frames, the ket state $|s\nu\rangle$ is given in a common rest frame. Its decay into $\gamma + \pi^0$ is nevertheless most efficiently described in the helicity basis, as shown in (2.3). So we see that we have adopted here a *mixture* of canonical and helicity prescriptions for decay amplitudes. The overall decay amplitude which includes $s \to s_1 + \pi_2$ is, absorbing N_s and N_{ℓ_3} into appropriate V's,

$$\begin{aligned} A^J_{M\,\nu_1} = V_{Jj} &\Bigg\{ \sum_{L\,m_j} N_L\, K^J_L\, (LM_L\, jm_j|JM)\; d^L_{M_L\,0}(\theta_0) \\ &\times (-)^{m_j-\nu} \sum_{\ell\,\nu} N_\ell\, G^j_\ell\, (\ell\, m\, s\nu|jm_j)\; d^\ell_{m\,0}(\theta_c) \Bigg\} f^{\,s}_{\nu_1}\, D^{s\,*}_{\nu\,\nu_1}(\phi_1, \theta_1, 0) \\ +V_{Jj'} &\Bigg\{ \sum_{L'\,m_j'} N_{L'}\, K^J_{L'}\, (L'M_L'\, j'm_j'|JM)\; d^{L'}_{M_L'\,0}(\theta_0') \\ &\times (-)^{m_j'-\nu} \sum_{\ell\,\nu} N_\ell\, G^{j'}_\ell\, (\ell\, m\, s\nu|j'm_j')\; d^\ell_{m\,0}(\theta_c') \Bigg\} f^{\,s}_{\nu_1}\, D^{s\,*}_{\nu\,\nu_1}(\phi_1, \theta_1, 0) \\ &+V_{J\ell_3} \Bigg\{ \sum_{\lambda_3} E^J_{\nu\,\lambda_3}\, d^{\ell_3}_{\lambda_3\,0}(\theta_3) \Bigg\} f^{\,s}_{\nu_1}\, D^{s\,*}_{\nu\,\nu_1}(\phi_1, \theta_1, 0) \end{aligned} \tag{5.14}$$

where $\nu = M + \lambda_3$ in the third term. This is to be compared with (3.21).

The amplitude for $N\pi_1\pi_1'$ systems is, from (5.14), absorbing N_{ℓ_3} into $V_{J\ell_3}$,

$$
\begin{aligned}
A^J_{M\,\nu} =& V_{Jj} \left\{ \sum_{L\,m_j} N_L\, K^J_L\, (LM_L\, jm_j|JM)\; d^L_{M_L\,0}(\theta_0) \right. \\
& \qquad\qquad \left. \times(-)^{m_j-\nu} \sum_{\ell} N_\ell\, G^j_\ell\, (\ell\, m\, s\nu|jm_j)\; d^\ell_{m\,0}(\theta_c) \right\} \\
& + V_{Jj'} \left\{ \sum_{L'\,m_j'} N_{L'}\, K^J_{L'}\, (L'M_L'\, j'm_j'|JM)\; d^{L'}_{M_L'\,0}(\theta_0') \right. \\
& \qquad\qquad \left. \times(-)^{m_j'-\nu} \sum_{\ell} N_\ell\, G^{j'}_\ell\, (\ell\, m\, s\nu|j'm_j')\; d^\ell_{m\,0}(\theta_c') \right\} \\
& + V_{J\ell_3} \left\{ \sum_{\lambda_3} E^J_{\nu\,\lambda_3}\, d^{\ell_3}_{\lambda_3\,0}(\theta_3) \right\}
\end{aligned}
\tag{5.15}
$$

which is to be comapred with (4.1).

6 Conclusions

The purpose of this note has been to show how one should treat the decay $\omega \to \gamma + \pi^0$, when it is observed through more than one sequential decays. The general solution requires introduction of additional sets of Euler angles, applied to ω *before* it is allowed to decay (this is because the helicity-coupling amplitudes $f^s_{\lambda_1}$ and the accompanying D-functions both depend on the photon helicity λ_1). In his thesis, Giarritta seems to imply that a general solution requires introduction of a third angle in the D-functions. It has been shown in this note that this is not the case.

The extra rotations are required because the γ helicity ($\lambda_1 = \pm 1$) is an 'external variable' (even though it is eventually summed over outside of the overall amplitudes squared), and hence it needs to be evaluated in a single frame. The reason we do not need this extra step, e.g. for the decay $\rho \to \pi\pi$, is that the decay products are both spinless. One recalls that the decay amplitude for $\omega \to 3\pi$ is formally identical to $\rho \to 2\pi$, because its 'helicity-coupling amplitude' are $F_\pm = 0$ and $F_0 \neq 0$, and so the ω helicities do not appear in the amplitudes (see Section 6, ref. [1]). This is simply an accident of the fact that we have $J^P = 1^-$ for the ω. If the ω had been $J^P = 1^+$, then the nonzero 'helicity-coupling amplitude' would have been $F_\pm \neq 0$ and $F_0 = 0$ and so the ω helicities would have appeared as 'external' variables.

An analysis of $N\pi\pi$ systems in which there are two different $N\pi$ isobars requires a similar treatment.

We have shown that a better treatment of the nonzero spins in the final states is to employ the canonical prescription for decay amplitudes.

Appendix A:
Two-Body decays in Helicity Formalism

We start with a decay amplitude in helicity formalism and use it to derive the recoupling coefficient between the rotationally invariant decay amplitudes in helicity and canonical formalism. See Section 4 of Ref.[1] for a standard treatment of this problem; our purpose here is to introduce a new set of notations which have been employed in Section 3 of this note, and to show its efficacy in dealing with helicity states defined in different coordinate systems.

Define

$$A_M^{J\lambda_1\lambda_2}(\Omega) = \sqrt{\frac{2J+1}{4\pi}}\, F^J_{\lambda_1\,\lambda_2} D^{J\,*}_{M\lambda}(\phi,\theta,0) \;\equiv\; {}_h\langle s_1\lambda_1\; s_2\; -\lambda_2|\psi\rangle \tag{6.1}$$

where $\lambda = \lambda_1 - \lambda_2$. With $R = R(\phi,\theta,0)$, we observe

$$|s\lambda\rangle_h = R\,|s\lambda\rangle_0 = \sum_\nu |s\nu\rangle_0\; {}_0\langle s\nu|R\,|s\lambda\rangle_0 = \sum_\nu D^s_{\nu\,\lambda}(R)\,|s\nu\rangle_0 \tag{6.2}$$

where $|s\lambda\rangle_0$ is a helicity state defined in the original coordinate system (x_0, y_0, z_0), i.e. $\vec{p}_1 - \vec{p}_2$ is along $\hat{z}_0$. and $|s\lambda\rangle_h$ is a helicity state defined in the helicity coordinate system (x_h, y_h, z_h) [see Fig. 1]. The decay amplitude in the canonical formalism [see Appendix B] is

$$\begin{aligned}
A^J_{M\,\nu_1\nu_2}(\Omega) &\equiv {}_0\langle s_1\nu_1\; s_2\nu_2|\psi\rangle = \sum_{\lambda_1\,\lambda_2} {}_0\langle s_1\nu_1|s_1\lambda_1\rangle_h\; {}_0\langle s_2\nu_2|s_2\; -\lambda_2\rangle_h\; {}_h\langle s_1\lambda_1\; s_2\; -\lambda_2|\psi\rangle \\
&= \sqrt{\frac{2J+1}{4\pi}} \sum_{\lambda_1\,\lambda_2} F^J_{\lambda_1\,\lambda_2} D^{J\,*}_{M\lambda}(R)\; D^{s_1}_{\nu_1\,\lambda_1}(R)\, D^{s_2}_{\nu_2\,-\lambda_2}(R) \\
&= \sqrt{\frac{2J+1}{4\pi}} \sum_{\lambda_1\,\lambda_2} F^J_{\lambda_1\,\lambda_2} \sum_\ell \left(\frac{2\ell+1}{2J+1}\right) (\ell\, m\; S\nu|JM)\,(s_1\nu_1\; s_2\nu_2|S\nu) \\
&\qquad\qquad \times (\ell 0\, S\lambda|J\lambda)\,(s_1\lambda_2 s_2\, -\lambda_2|S\lambda)\, D^{\ell\,*}_{m\,0}(R) \\
&= \sum_\ell \sqrt{\frac{2\ell+1}{4\pi}}\, G^J_{\ell S}\,(\ell\, m\; S\nu|JM)\,(s_1\nu_1\; s_2\nu_2|S\nu)\, D^{\ell\,*}_{m\,0}(R) \\
&= \sum_\ell G^J_{\ell S}\,(\ell\, m\; S\nu|JM)\,(s_1\nu_1\; s_2\nu_2|S\nu)\, Y^\ell_m(\Omega)
\end{aligned} \tag{6.3}$$

where

$$G^J_{\ell S} = \left(\frac{2\ell+1}{2J+1}\right)^{1/2} \sum_{\lambda_1\,\lambda_2} F^J_{\lambda_1\,\lambda_2}\,(\ell 0\, S\lambda|J\lambda)\,(s_1\lambda_2 s_2\, -\lambda_2|S\lambda) \tag{6.4}$$

We have derived the standard formula for decay amplitudes in canonical formalism.

Appendix B: Two-Body decays in Canonical Formalism

We give a brief description of the decay amplitudes in canonical formalism. Consider a two-body state $|s_1 m_1\rangle + |s_2 m_2\rangle$ with momentum $\vec{p}$ in the RF

$$\begin{aligned} |\vec{p}\, m_1;\ -\vec{p}\, m_2\rangle &= U[L(\vec{p}\,)]\, |s_1 m_1\rangle U[L(-\vec{p}\,)]\, |s_2 m_2\rangle \\ &= \frac{1}{a}\, |\Omega\, m_1\, m_2\rangle, \qquad\qquad a = \frac{1}{4\pi}\sqrt{\frac{p}{w}} \end{aligned} \tag{6.5}$$

where $\Omega = (\theta, \phi)$ describes the direction of $\vec{p}$ in the RF, and m_1 and m_2 are the z-compoents of spin in the canonical quantization. The relevant boost operators have been denoted by $U[L(\pm\vec{p}\,)]$. The decay amplitude for $|JM\rangle \to |s_1 m_1\rangle + |s_2 m_2\rangle$ is, in the JRF,

$$\begin{aligned} A^J_{M m_1 m_2}(\Omega) &= \langle \vec{p}\, m_1;\ -\vec{p}\, m_2 | \mathcal{M} | JM\rangle \\ &= 4\pi \left(\frac{w}{p}\right)^{1/2} \langle \Omega\, m_1\, m_2 | \mathcal{M} | JM\rangle \\ &= 4\pi \left(\frac{w}{p}\right)^{1/2} \sum_{\ell S} \langle \Omega\, m_1\, m_2 | JM\ell\, S\rangle \langle JM\ell\, S | \mathcal{M} | JM\rangle \\ &= \sum_{\ell S} G^J_{\ell S}\, (\ell m\, S m_s | JM)(s_1 m_1\, s_2 m_2 | S m_s)\, Y^\ell_m(\Omega) \end{aligned} \tag{6.6}$$

where $m_s = m_1 + m_2$ and $m = M - m_s$. The ℓS-coupling amplitude is given by

$$G^J_{\ell S} = 4\pi \left(\frac{w}{p}\right)^{1/2} \langle JM\ell\, S | \mathcal{M} | JM\rangle \tag{6.7}$$

So the decay amplitude is

$$A^J_{M m_1 m_2}(\Omega) = \sum_{\ell S} \sqrt{\frac{2\ell+1}{4\pi}}\, G^J_{\ell S}\, (\ell m\, S m_s | JM)(s_1 m_1\, s_2 m_2 | S m_s)\, D^{\ell\,*}_{m0}(\phi, \theta, 0) \tag{6.8}$$

It is instructive to compare (6.8) with the decay amplitude given in helicity formalism

$$A^{J\lambda_1\lambda_2}_M(\Omega) = \sqrt{\frac{2J+1}{4\pi}}\, F^J_{\lambda_1\,\lambda_2} D^{J\,*}_{M\lambda}(\phi, \theta, 0) \tag{6.9}$$

where $\lambda = \lambda_1 - \lambda_2$ and

$$F^J_{\lambda_1\,\lambda_2} = \sum_{\ell S} \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} G^J_{\ell S}\, (\ell 0\, S\lambda | J\lambda)\, (s_1 \lambda_2 s_2\, {-\lambda_2} | S\lambda) \tag{6.10}$$

so that

$$A^{J\lambda_1\lambda_2}_M(\Omega) = \sum_{\ell S} \sqrt{\frac{2\ell+1}{4\pi}}\, G^J_{\ell S}\, (\ell 0\, S\lambda | J\lambda)\, (s_1 \lambda_2 s_2\, {-\lambda_2} | S\lambda) D^{J\,*}_{M\lambda}(\phi, \theta, 0) \tag{6.11}$$

Comparing this to (6.8), we see that

$$A^J_{Mm_1m_2}(\Omega) = A^{J\lambda_1\lambda_2}_M(\Omega), \qquad \text{if} \quad \phi = \theta = 0 \tag{6.12}$$

with $m_1 = \lambda_1$ and $m_2 = -\lambda_2$. Note, in addition, that the D-functions in helicity formalism couple directly to J with the second subscript depending on $\lambda = \lambda_1 - \lambda_2$, whereas the D-functions in the canonical formulation couple to ℓ with the second subscript set to zero. Three rotational invariants $\{J\ell\, S\}$ in canonical formalism have been transformed into three rotational invariants $\{J\lambda_1\, \lambda_2\}$ in helicity formalism.

Appendix C: A Detailed Example for $\bar{p}p \to \omega + \pi_1 + \pi_1'$

Consider an example given in Fig. 2, where two parallel sequential decays of (3.2) are illustrated. In the overall CM system, we begin by setting

$$(\hat{x}_0, \hat{y}_0, \hat{z}_0): \quad \hat{x}_0 = (1,0,0), \quad \hat{y}_0 = (0,1,0), \quad \text{and} \quad \hat{z}_0 = (0,0,1)$$

The illustration in Fig. 2 corresponds the 4-momenta given by

4-mom	E	p_x	p_y	p_z
$\bar{p}p$	1.7699	0	0	0
$\omega\,(s)$	0.9857	0	0	0.6000
π_1	0.3156	0.2000	0	-0.2000
π_1'	0.4686	-0.2000	0	-0.4000

(6.13)

with the relevant angles given by

$$\Omega_0 = (\theta_0, 0^\circ),\ \theta_0 = 26.5650^\circ, \qquad \Omega_0' = (\theta_0', 180^\circ),\ \theta_0' = 45.000^\circ$$
$$\Omega = (\theta, 180^\circ),\ \theta = 51.8542^\circ, \qquad \Omega' = (\theta', 180^\circ),\ \theta' = 60.8024^\circ \tag{6.14}$$

In the ωRF, the relevant 4-momenta are given by

4-mom	E	p	θ	ϕ	p_x	p_y	p_z
$\omega\,(s)$	0.7820	0	0	0	0	0	0
$\gamma\,(s_1)$	0.3785	0.3785	30°	60°	0.0946	0.1639	0.3278
$\pi^0\,(\pi_2)$	0.4035	0.3785	150°	240°	-0.0946	-0.1639	-0.3278

(6.15)

all measured in the coordinate system $(\hat{x}_0, \hat{y}_0, \hat{z}_0)$.

We first bring γ and π^0 into the overall CM system ($\bar{p}p$ RF) via pure, time-like Lorentz transformations. Next we Lorentz-transform ω, π_1, π_1', γ and π^0 into jRF (j'RF) and then into ωRF (ω'RF), so that the final coordinate systems are $(\hat{x}_2, \hat{y}_2, \hat{z}_2)$ $[\,(\hat{x}_2', \hat{y}_2', \hat{z}_2')\,]$. We use the helicity coordinate system of Fig. 1 for each stage. The results are

$$\hat{x}_2 = (-0.9042, 0, -0.4272), \quad \hat{y}_2 = (0, -1, 0), \quad \hat{z}_2 = (-0.4272, 0, +0.9042)$$
$$\hat{x}_2' = (+0.9622, 0, -0.2723), \quad \hat{y}_2' = (0, +1, 0), \quad \hat{z}_2' = (+0.2723, 0, +0.9622) \tag{6.16}$$

In these coordinate systemsm, the direction of γ is given by

$$\begin{array}{c|cc} \gamma \text{ direction} & \theta_1^{(2)} & \phi_1^{(2)} \\ \hline \Omega_1 \text{ in } (\hat{x}_2, \hat{y}_2, \hat{z}_2) & 44.8459^\circ & -142.1188^\circ \\ \Omega_1 \text{ in } (\hat{x}'_2, \hat{y}'_2, \hat{z}'_2) & 25.8290^\circ & 83.6494^\circ \end{array} \tag{6.17}$$

while the direction of γ and its momentum with respect to the original coordinate system is

$$\begin{array}{c|ccccc} \gamma \text{ direction} & \theta_1^{(0)} & \phi_1^{(0)} & p_x^{(0)} & p_y^{(0)} & p_z^{(0)} \\ \hline \Omega_1 \text{ in } (\hat{x}_0, \hat{y}_0, \hat{z}_0) & 28.4982^\circ & 65.1666^\circ & 0.0758 & 0.1639 & 0.3326 \\ \Omega'_1 \text{ in } (\hat{x}_0, \hat{y}_0, \hat{z}_0) & 31.4652^\circ & 56.0532^\circ & 0.1103 & 0.1639 & 0.3228 \end{array} \tag{6.18}$$

These values are close to but not equal to the original values [see (6.15)] given by

$$\Omega_1 = (\theta_1, \phi_1), \quad \theta_1 = 30^\circ, \quad \text{and} \quad \phi_1 = 60^\circ$$

which is in fact the dirction of γ measured in $(\hat{x}_0, \hat{y}_0, \hat{z}_0)$ (or, equivalently, the γ 4-momentum in the overall CM system has been Lorentz-transformed directly into the ωRF—without going through the jRF or the j'RF). Lorentz transformations in this example is confined to the zx-plane; so the y-components remain invariant under the transformations. Comparing the z- and x-components of the γ momenta in (6.15) and (6.18), we observe a small but finite rotation in the zx-plane (or around the y-axis)

$$\Delta\beta = -3.2575^\circ, \qquad \Delta\beta' = +2.7650^\circ \tag{6.19}$$

The rotation around the y-axis which takes $(\hat{x}_2, \hat{y}_2, \hat{z}_2)$ into the original coordinate system $(\hat{x}_0, \hat{y}_0, \hat{z}_0)$ and again $(\hat{x}'_2, \hat{y}'_2, \hat{z}'_2)$ into $(\hat{x}_0, \hat{y}_0, \hat{z}_0)$ are

$$(\alpha,\, \beta,\, \gamma) = (180^\circ,\ 25.2891^\circ,\ 0^\circ) \quad \text{and} \quad (\alpha',\beta',\, \gamma') = (0^\circ,\ -15.8024^\circ,\ 0^\circ) \tag{6.20}$$

we see that, from (6.14),

$$-\theta_0 + \theta - \beta = 0, \quad \text{and} \quad -\theta'_0 + \theta' + \beta' = 0 \tag{6.21}$$

Finally, we need to know the direction of s (or ω) with respect to the original coordinate system $(\hat{x}_0, \hat{y}_0, \hat{z}_0)$. There are two ways of measuring the angles; $\Omega_c = (\theta_c, \phi_c)$ which describe the direction of ω in the jRF and $\Omega'_c = (\theta'_c, \phi'_c)$ which describe the direction of ω in the j'RF

$$\begin{aligned} \Omega_c &= (\theta_c, \phi_c), \quad \theta_c = 25.2891^\circ, \quad \text{and} \quad \phi_c = 180^\circ \\ \Omega'_c &= (\theta'_c, \phi'_c), \quad \theta'_c = 15.8024^\circ, \quad \text{and} \quad \phi'_c = 0^\circ \end{aligned} \tag{6.22}$$

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References

[1] S. U. Chung, 'Spin Formalisms,' CERN Yellow Report, CERN 71-8 (1971); Updated Version, `http://cern.ch/suchung/`.

[2] P. Giarritta, 'Search for New mesons in $p\bar{p}$ Annihilation into $\omega\pi^0\pi^0$ and $\omega\eta\pi^0$,' PhD Thesis, University of Zurich (2000) [Thesis Adviser: C. Amsler].

[3] M. E. Rose, 'Elementary Theory of Angular Momentum,' John Wiley & Sons, Inc., Eq. (4.16) and Eq. (4.17).